The Motion of a Solid in Infinite Liquid under no Forces.

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The theory is sketched out in Thomson and Tait's "Natural Philosophy," and a complete solution for a solid of revolution is given in Kirchhoff's "Vorlesungen über Mathematische Physik," IX; the treatment is developed at length in the "Motion of a Solid in a Liquid," by Dr. Thomas Craig (Van Nostrand, 1878), and in Halphen's "Fonctions elliptiques," t. II, chap. IV, and the subject in general has attracted the attention of a large number of writers, for which the Fortschritte der Mathematik may be consulted.

The object of the present paper is to examine closely the elliptic function expression of all the dynamical quantities involved, and to explore the analytical field by working out completely the simplest Pseudo-Elliptic cases to serve as landmarks, utilizing for this purpose the analysis developed in the paper on "Pseudo-Elliptic Integrals and their Dynamical Applications," Proc. London Math. Society, vol. XXV, and carrying this out in continuation of the manner employed for a similar purpose in the papers on the "Dynamics of a Top" and on the "Associated Motion of a Top and of a Body under no Forces," Proc. London Math. Society, vols. XXVI, XXVII.

The lectures delivered last year by Professor Klein at Princeton University have placed the analytical treatment of the motion of the top, and of Jacobi's two allied motions *à la Poinsot*, in a much clearer light and in a more elegant manner; a similar treatment of the present problem will certainly prove equally valuable, but meanwhile the special cases, developed at length here, will serve as oases, so to speak, in the infinite region of the general elliptic function solution.

Simple experimental illustrations of the motion can be observed in the evolutions of a plate or coin or bubble in water, or of a disc of paper or cardboard in the air, as well as in the motion of a projectile or torpedo.

1. The notation employed is that given in Basset's "Hydrodynamics," vol. I, Appendix III, and also in the "Applications of Elliptic Functions," p. 342; the body may for simplicity be taken as a smooth homogeneous solid of revolution, having component linear and angular velocities u, v, w and p, q, r; and now the total kinetic energy T of the body and of the surrounding infinite frictionless liquid stirred up by the motion of the body is given by an expression of the form

$$T = \tfrac{1}{2} P(u^2 + v^2) + \tfrac{1}{2} Rw^2 + \tfrac{1}{2} A(p^2 + q^2) + \tfrac{1}{2} Cr^2, \tag{A}$$

where P, R, A, C are constants depending on the shape of the body and on the density of the solid and liquid.

The more general form of T assumed by Halphen, due to certain modifications in the shape of the body, which still leads to Elliptic Function solutions, may be considered separately in its modification of the results, as also the effect of the *circulation* of the liquid, which may exist with a ring-shaped or perforated body.

To realize practically the condition that no external forces act upon the body, even in a field of gravity, we may make the density of the body and of the liquid the same, so that the buoyancy and weight cancel, and the apparent weight is zero, as in the case of a fish and a submarine boat or torpedo; and now the Hamiltonian equations of motion lead to

$$P\frac{du}{dt} - rPv + qRw = 0, \tag{1}$$

$$P\frac{dv}{dt} - pRw + rPu = 0, \tag{2}$$

$$R\frac{dw}{dt} - qPu + pPv = 0, \tag{3}$$

$$A\frac{dp}{dt} - rAq + qCr - wPv + vRw = 0, \tag{4}$$

$$A\frac{dq}{dt} - pCr + rAp - uRw + wPu = 0, \tag{5}$$

$$C\frac{dr}{dt} - qAp + pAq - vPu + uPv = 0; \tag{6}$$

some of the equations being capable of obvious simplifications.

2. To make the memoir complete, it will be advisable to repeat to a certain extent the ordinary treatment; thus, multiplying (1) by Pu, (2) by Pv, (3) by Rw, and adding,

$$P^2u\frac{du}{dt}+P^2v\frac{dv}{dt}+R^2w\frac{dw}{dt}=0, \tag{7}$$

and integrating,

$$P^2(u^2+v^2)+R^2w^2=F^2, \tag{B}$$

where the constant F represents the resultant *linear momentum* of the system.

Similarly it can be shown that

$$AuPp+AvPq+CwRr=G, \tag{C}$$

where G is a constant, representing the resultant *angular momentum* of the system.

From equations (A) and (B),

$$\begin{aligned} A(p^2+q^2) &= 2T-Cr^2-Rw^2-P(u^2+v^2) \\ &= 2T-Cr^2-\frac{F^2}{R}+F^2\left(\frac{1}{R}-\frac{1}{P}\right)(F^2-R^2w^2), \end{aligned}$$

and from equation (3),

$$\begin{aligned} R^2\frac{dw^2}{dt^2} &= P^2(uq-vp)^2 \\ &= P^2(u^2+v^2)(p^2+q^2)-P^2(up+vq)^2 \\ &= \frac{F^2}{A}\left(\frac{1}{R}-\frac{1}{P}\right)(F^2-R^2w^2)^2 \\ &\quad +\left(2T-Cr^2-\frac{F^2}{R}\right)\frac{F^2-R^2w^2}{A}-\left(\frac{G-CwRr}{A}\right)^2, \end{aligned} \tag{D}$$

thus determining w or Rw as an elliptic function of t.

3. By the Principles of the Conservation of Energy and Momentum, T will remain constant, while F will represent a constant linear momentum in a fixed direction, Oz suppose; so that denoting by $\gamma_1, \gamma_2, \gamma_3$ the cosines of the angles between Oz and the axes OA, OB, OC fixed in the body,

$$Pu=F\gamma_1,\quad Pv=F\gamma_2,\quad Rw=F\gamma_3; \tag{8}$$

and, introducing Euler's angles θ, ϕ, ψ,

$$\gamma_1 = -\sin\theta\cos\phi,\quad \gamma_2 = \sin\theta\sin\phi,\quad \gamma_3 = \cos\theta; \tag{9}$$

$$\begin{aligned} p &= \sin\phi\,\dot\theta - \sin\theta\cos\phi\,\dot\psi, \\ q &= \cos\phi\,\dot\theta + \sin\theta\sin\phi\,\dot\psi, \\ r &= \dot\phi + \cos\theta\,\dot\psi, \end{aligned} \tag{10}$$

so that

$$\begin{aligned} P(up + vq) &= F\sin\theta(-p\cos\phi + q\sin\phi) \\ &= F\sin^2\theta\,\frac{d\psi}{dt}, \end{aligned} \tag{11}$$

or

$$\frac{d\psi}{dt} = \frac{G - CrF\cos\theta}{AF\sin^2\theta}. \tag{12}$$

Split up into two partial fractions,

$$\frac{d\psi}{dt} = \frac{d\psi_1}{dt} + \frac{d\psi_2}{dt}, \tag{13}$$

where

$$\begin{aligned} \frac{d\psi_1}{dt} &= \frac{G - CrF}{2AF}\,\frac{1}{1-\cos\theta}, \\ \frac{d\psi_2}{dt} &= \frac{G + CrF}{2AF}\,\frac{1}{1+\cos\theta}; \end{aligned} \tag{14}$$

and then

$$\begin{aligned} \frac{d\phi}{dt} &= r - \cos\theta\,\frac{d\psi}{dt} \\ &= \left(1 - \frac{C}{A}\right)r - \frac{d\psi_1}{dt} + \frac{d\psi_2}{dt}. \end{aligned} \tag{15}$$

4. Writing z for $\cos\theta$, or Fz for Rw in equation (D), then

$$\frac{dz^2}{dt^2} = n^2\left\{a(z^2-1)^2 - \left(2T - Cr^2 - \frac{F^2}{R}\right)\frac{z^2-1}{An^2} - \left(\frac{CrFz - G}{AnF}\right)^2\right\}, \tag{16}$$

where

$$an^2 = \frac{F^2}{A}\left(\frac{1}{R} - \frac{1}{P}\right); \tag{17}$$

and according as the body is prolate or oblate, so is $P > R$ or $P < R$, or an^2 is positive or negative.

To distinguish these two cases, we take $a = +1$ for prolate bodies and $a = -1$ for oblate bodies; so that

$$n^2 = \frac{F^2}{A}\left(\frac{1}{R} \sim \frac{1}{P}\right), \tag{18}$$

and now
$$\frac{dz}{dt} = n\sqrt{Z}, \tag{19}$$
where
$$Z = a(z^2-1)^2 - \left(2T - Cr^2 - \frac{F^2}{R}\right)\frac{z^2-1}{An^2} - \left(\frac{CrFz - G}{AnF}\right)^2 \tag{20}$$
and
$$\frac{d\psi}{dz} = \frac{G - Crz}{AnF(1-z^2)\sqrt{Z}}. \tag{21}$$

Following the notation employed in the discussion of the motion of a top, it is convenient to put
$$2T - Cr^2 - \frac{F^2}{R} = An^2D, \tag{22}$$
and
$$D - \frac{G^2 - C^2r^2F^2}{A^2n^2F^2} = E, \tag{23}$$
so that
$$Z = a(z^2-1)(z^2-1-aD) - \left(\frac{CrFz - G}{AnF}\right)^2, \tag{24}$$
$$Z = a(z^2-1)(z^2-1-aE) - \left(\frac{Gz - CrF}{AnF}\right)^2. \tag{25}$$

5. Starting now with the general elliptic integral of the first kind,
$$u = \int\frac{dz}{\sqrt{Z}}, \tag{26}$$
where the quartic
$$Z = az^4 + 4bz^3 + 6cz^2 + 4dz + e, \tag{27}$$
and u denotes the elliptic argument (a different use of the letter u to that employed previously to denote a component velocity; these two meanings will not however be found to clash); we make use of a theorem, supposed to be due to Weierstrass, but first employed by Biermann in his Dissertation, "Problemata quædam mechanica functionum ellipticarum ope soluta," 1865, which asserts that we can put
$$\wp(u_1 \pm u_2) = \frac{F(z_1, z_2) \mp \sqrt{Z_1}\sqrt{Z_2}}{2(z_1 - z_2)^2}, \tag{E}$$
where
$$F(z_1, z_2) = az_1^2z_2^2 + 2bz_1z_2(z_1+z_2) + c(z_1^2 + 4z_1z_2 + z_2^2) + 2d(z_1+z_2) + e, \tag{28}$$
the *second polar* of Z; u_1 and u_2 denoting the elliptic arguments corresponding to z_1 and z_2.

6. When the roots e_α, e_β, e_γ of the discriminating cubic

$$4e^3 - g_2 e - g_3 = 0 \tag{29}$$

are known, in which g_2 and g_3 are the *quadrinvariant* and *cubinvariant* of the quartic Z, given by

$$g_2 = ae - 4bd + 3c^2, \tag{30}$$

$$g_3 = ace + 2bcd - ad^2 - eb^2 - c^3, \tag{31}$$

the above theorem (E) can be replaced by

$$\wp(u_1 \pm u_2) - e_\alpha$$
$$= a\left\{\frac{\sqrt{(z_1 - z_0 . z_1 - z_\alpha)}\sqrt{(z_2 - z_\beta . z_2 - z_\gamma)} \mp \sqrt{(z_2 - z_0 . z_2 - z_\alpha)}\sqrt{(z_1 - z_\beta . z_1 - z_\gamma)}}{2(z_1 - z_2)}\right\}^2, \tag{32}$$

where z_0, z_α, z_β, z_γ denote the roots of the quartic $Z = 0$, so that

$$Z = a(z - z_0)(z - z_\alpha)(z - z_\beta)(z - z_\gamma). \tag{33}$$

With the form of Z in (20),

$$z_0 + z_\alpha + z_\beta + z_\gamma = 0, \tag{34}$$

$$a(z_0 + z_\alpha)(z_\beta + z_\gamma) + a(z_0 z_\alpha + z_\beta z_\gamma) = -2a - D - \frac{C^2 r^2}{A^2 n^2}, \tag{35}$$

$$a z_0 z_\alpha z_\beta z_\gamma = a + D - \frac{G^2}{A^2 n^2 F^2}. \tag{36}$$

Thence, denoting by v_1 and v_2 the values of the elliptic argument corresponding to

$$z_1 = +1 \text{ and } z_2 = -1,$$

$$Z_1 Z_2 = -\frac{C^2 r^2 F^2 - G^2}{A^2 n^2 F^2}, \tag{37}$$

$$\wp(v_1 \pm v_2) - e_\alpha$$
$$= \tfrac{1}{8} a\{1 - (z_0 + z_\alpha)(z_\beta + z_\gamma) + z_0 z_\alpha + z_\beta z_\gamma + z_0 z_\alpha z_\beta z_\gamma\} \pm \tfrac{1}{8}\frac{C^2 r^2 F^2 - G^2}{A^2 n^2 F^2}$$
$$= \tfrac{1}{8}\left\{a + 2a(z_0 + z_\alpha)^2 - 2a - D - \frac{C^2 r^2}{A^2 n^2} + a + D - \frac{G^2}{A^2 n^2 F^2} \pm \frac{C^2 r^2 F^2 - G^2}{A^2 n^2 F^2}\right\}; \tag{38}$$

or

$$\wp(v_1 + v_2) - e_\alpha = \tfrac{1}{4} a (z_0 + z_\alpha)^2 - \frac{G^2}{4A^2 n^2 F^2}, \tag{39}$$

$$\wp(v_1 - v_2) - e_\alpha = \tfrac{1}{4} a (z_0 + z_\alpha)^2 - \frac{C^2 r^2}{4A^2 n^2}; \tag{40}$$

so that

$$\tfrac{1}{4} a (z_0 + z_\alpha)^2 = \frac{G^2}{4A^2n^2F^2} + \wp(v_1 + v_2) - e_\alpha, \tag{41}$$

$$= \frac{C^2r^2}{4A^2n^2} + \wp(v_1 - v_2) - e_\alpha. \tag{42}$$

7. It is convenient to employ the notation of Darboux in the corresponding motion of the top (Proc. London Math. Society, vol. XXVII), and to put

$$\frac{G}{AnF} = \frac{2L}{M}, \quad \frac{Cr}{An} = \frac{2B}{M}; \tag{43}$$

so that

$$Z = a(z^2 - 1)(z^2 - 1 - aD) - 4\left(\frac{Bz - L}{M}\right)^2, \tag{44}$$

$$= a(z^2 - 1)(z^2 - 1 - aE) - 4\left(\frac{Lz - B}{M}\right)^2, \tag{45}$$

and

$$\frac{d\psi}{dz} = 2\frac{L - Bz}{M(1 - z^2)\sqrt{Z}}. \tag{46}$$

Also

$$\wp u - e_\alpha = \frac{s - s_\alpha}{M^2}, \tag{47}$$

where M is a *homogeneity factor* to be determined in the sequel, and s is a new variable defined by

$$\frac{u}{M} = \int \frac{ds}{\sqrt{S}} = \frac{1}{M}\int \frac{dz}{\sqrt{Z}}, \tag{48}$$

where

$$S = 4s(s + x)^2 - \{(y + 1)s - xy\}^2, \tag{49}$$

x and y being the quantities defined by Halphen in his "Fonctions elliptiques," t. I, p. 103; and when resolved into factors, we put

$$S = 4(s - s_\alpha)(s - s_\beta)(s - s_\gamma), \text{ or } 4(s - s_1)(s - s_2)(s - s_3). \tag{50}$$

8. Now putting

$$v_1 + v_2 = v, \tag{51}$$

(again a different signification of v to that employed at the outset) and denoting by σ or $s(v)$ the value of s or $s(u)$ corresponding to $u = v$, equation (41) becomes

$$\begin{aligned} \tfrac{1}{4} aM^2 (z_0 + z_\alpha)^2 &= L^2 + \sigma - s_\alpha \\ &= aN_\alpha^2, \end{aligned} \tag{52}$$

suppose; so that

$$\begin{aligned} \tfrac{1}{2} M(z_0 + z_1) &= N_1, \\ \tfrac{1}{2} M(z_0 + z_2) &= N_2, \\ \tfrac{1}{2} M(z_0 + z_3) &= N_3. \end{aligned} \tag{53}$$

Taking $s_1 > s_2 > s_3$, and employing relation (34), we find that when $a = +1$, as for a prolate solid,

$$\begin{aligned} Mz_0 &= -N_1 + N_2 + N_3, \\ Mz_3 &= +N_1 - N_2 + N_3, \\ Mz_2 &= +N_1 + N_2 - N_3, \\ Mz_1 &= -N_1 - N_2 - N_3; \end{aligned} \tag{54}$$

while for $a = -1$, and an oblate solid,

$$\begin{aligned} Mz_3 &= +N_1 + N_2 - N_3, \\ Mz_2 &= +N_1 - N_2 + N_3, \\ Mz_1 &= -N_1 + N_2 + N_3, \\ Mz_0 &= -N_1 - N_2 - N_3. \end{aligned} \tag{55}$$

9. Now, rewriting the expressions for Z, in this new notation,

$$\begin{aligned} Z &= a(z-z_0)(z-z_\alpha)(z-z_\beta)(z-z_\gamma) \\ &= a\left\{z^2 - 2\frac{N_\alpha}{M}z + \frac{N_\alpha^2 - (N_\beta - N_\gamma)^2}{M^2}\right\}\left\{z^2 + 2\frac{N_\alpha}{M} + \frac{N_\alpha^2 - (N_\beta - N_\gamma)^2}{M^2}\right\} \\ &= a\left(z^4 - 2\frac{N_1^2 + N_2^2 + N_3^2}{M^2}z^2 + 8\frac{N_1N_2N_3}{M^3}z \right. \\ &\qquad \left. + \frac{N_1^4 + N_2^4 + N_3^4 - 2N_2^2N_3^2 - 2N_3^2N_1^2 - 2N_1^2N_2^2}{M^4}\right); \end{aligned} \tag{56}$$

so that, putting $z = \pm 1$ in (56) and (44), (45),

$$a\left(1 - 2\frac{N_1^2 + N_2^2 + N_3}{M^2} + 8\frac{N_1N_2N_3}{M^3} + \frac{N_1^4 + \dots - 2N_2^2N_3^2\dots}{M^4}\right) = -4\left(\frac{B-L}{M}\right)^2, \tag{57}$$

$$a\left(1 - 2\frac{N_1^2 + N_2^2 + N_3^2}{M^2} - 8\frac{N_1N_2N_3}{M^3} + \frac{N_1^4 + \dots - 2N_2^2N_3^2\dots}{M^4}\right) = -4\left(\frac{B+L}{M}\right)^2. \tag{58}$$

Subtracting,
$$16a\frac{N_1N_2N_3}{M^3} = 16\frac{BL}{M^2}$$

or
$$BLM = aN_1N_2N_3, \tag{59}$$

so that
$$\frac{Cr}{An} = \frac{2B}{M} = 2a\frac{N_1N_2N_3}{LM^2} \tag{60}$$

with
$$\frac{G}{AnF} = \frac{2L}{M}. \tag{61}$$

Multiplying equations (57) and (58) by a, and adding,

$$1-2\frac{N_1^2+N_2^2+N_3^2}{M^2}+\frac{N_1^4+N_2^4+N_3^4-2N_2^2N_3^2-2N_3^2N_1^2-2N_1^2N_2^2}{M^4}$$

$$=-4a\frac{B^2+L^2}{M^2}$$

$$=-4a\frac{N_1^2N_2^2N_3^2}{L^2M^4}-4a\frac{L^2}{M^2},$$

$$L^2M^4-2(N_1^2+N_2^2+N_3^2-2aL^2)L^2M^2$$
$$+(N_1^4+\ldots\ldots-2N_2^2N_3^2-\ldots\ldots)L^2+4aN_1^2N_2^2N_3^2=0,$$

$$L^2(M^2-N_1^2-N_2^2-N_3^2+2aL^2)^2=4(L^2-aN_1^2)(L^2-aN_2^2)(L^2-aN_3^2)$$
$$=4(s_1-\sigma)(s_2-\sigma)(s_3-\sigma)=-\Sigma, \tag{62}$$

where Σ denotes the value of S in (50) when $s=\sigma$; and thus

$$M^2=N_1^2+N_2^2+N_3^2-2aL^2-\frac{\sqrt{-\Sigma}}{L} \tag{63}$$

thus determining the homogeneity factor M.

10. Equating the coefficients of z^2,

$$2a+D+4\frac{B^2}{M^2}=2a+E+4\frac{L^2}{M^2}$$
$$=2a\frac{N_1^2+N_2^2+N_3^2}{M^2}; \tag{64}$$

so that

$$E=2a\frac{N_1^2+N_2^2+N_3^2-2aL^2-M^2}{M^2}$$
$$=2a\frac{\sqrt{-\Sigma}}{LM^2}; \tag{65}$$

and

$$D=2a\frac{\sqrt{-\Sigma}}{LM^2}+4\frac{L^2-B^2}{M^2}. \tag{66}$$

But otherwise, equating the coefficients of z^0,

$$a+D-4\frac{L^2}{M^2}=a+E-4\frac{B^2}{M^2}$$
$$=a\frac{N_1^4+\ldots\ldots-2N_2^2N_3^2\ldots\ldots}{M^4}, \tag{67}$$

$$a+D=\frac{4L^2(N_1^2+N_2^2+N_3^2-2aL^2)-4L\sqrt{-\Sigma}+a(N_1^4+\ldots\ldots-2N_2^2N_3^2\ldots\ldots)}{M^4} \tag{68}$$

and

$$N_1^2 + N_2^2 + N_3^2 = 3aL^2 + 3a\wp v, \tag{69}$$

$$N_1^4 + \ldots\ldots - 2N_2^2N_3^2 \ldots\ldots = -3L^4 - 6L^2M^2\wp v + 9M^4\wp^2 v - 2M^4\wp'' v; \tag{70}$$

so that

$$a + D = a\frac{(L^2+3M^2\wp v)^2 - 4aL\sqrt{(-\Sigma)} - 2M^4\wp'' v}{M^4}, \tag{71}$$

an expression analogous to that in equation (214), p. 581, Proc. London Math. Society, vol. XXVII.

We may also write

$$E = -2\frac{Mi\wp' v}{L}, \tag{72}$$

and then by analogy,

$$D = \quad 2\frac{Mi\wp' w}{B}. \tag{73}$$

where

$$w = v_1 - v_2, \tag{74}$$

again a different use of w.

11. Another resolution of the quartic Z can be given, by means of the elliptic functions of the argument v_3, which corresponds to the *infinite* value of z; namely,

$$\sqrt{(a)}(z - z_0) = \frac{-\wp' v_3}{\wp u - \wp v_3}, \tag{75}$$

$$\sqrt{a}\,(z - z_\alpha) = \frac{-\wp' v_3}{\wp u - \wp v_3}\,\frac{\wp u - e_\alpha}{\wp v_3 - e_\alpha}. \tag{76}$$

Differentiating,

$$\sqrt{(aZ)} = \sqrt{a}\,\frac{dz}{du} = \frac{\wp' u\,\wp' v_3}{(\wp u - \wp v_3)^2}$$

$$= \wp(u - v_3) - \wp(u + v_3), \tag{77}$$

and integrating,

$$\sqrt{(a)}\,z = \zeta(u + v_3) - \zeta(u - v_3) - \zeta 2v_3$$

$$= \tfrac{1}{2}\frac{\wp'(u - v_3) - \wp' 2v_3}{\wp(u - v_3) - \wp 2v_3}$$

$$= \tfrac{1}{2}\frac{\wp'(u - v_3) + \wp'(u + v_3)}{\wp(u - v_3) - \wp(u + v_3)}; \tag{78}$$

and squaring,

$$az^2 = \wp(u - v_3) + \wp(u + v_3) + \wp 2v_3. \tag{79}$$

12. Putting $u_1 = u_2 = u$ in formula (E) leads to Hermite transformation

$$\wp 2u = -\frac{H}{Z}, \tag{80}$$

where H is the Hessian of the quartic Z, namely,

$$H = (ac - b^2)z^4 + 2(ad - bc)z^3 + (ae + 2bd - 3c^2)z^2 + 2(be - cd)z + ce - d^2. \tag{81}$$

Since $b = 0$ in our form of the quartic Z (20), then corresponding to $z = \infty$,

$$\wp 2v_3 = -c, \tag{82}$$

or $6\wp 2v_3$ is the coefficient of $-z^2$ in Z; so that

$$\begin{aligned}\wp 2v_3 &= \tfrac{1}{3}a + \tfrac{1}{6}D + \tfrac{2}{3}\frac{B^2}{M^2} \\ &= \tfrac{1}{3}a + \tfrac{1}{6}E + \tfrac{2}{3}\frac{L^2}{M^2} \\ &= a\tfrac{1}{3}\frac{N_1^2 + N_2^2 + N_3^2}{M^2} \\ &= \frac{L^2}{M^2} + \wp(v_1 + v_2),\end{aligned} \tag{83}$$

or
$$\wp(v_1 + v_2) = \wp 2v_3 - \frac{L^2}{M^2}; \tag{84}$$

and, by analogy,

$$\wp(v_1 - v_2) = \wp 2v_3 - \frac{B^2}{M^2}; \tag{85}$$

or
$$\wp(v_1 + v_2) = \tfrac{1}{3}a + \tfrac{1}{6}E - \tfrac{1}{3}\frac{L^2}{M^2}, \tag{86}$$

$$\wp(v_1 - v_2) = \tfrac{1}{3}a + \tfrac{1}{6}D - \frac{B^2}{M^2}; \tag{87}$$

while from (72) and (73),

$$\begin{aligned} i\wp'(v_1 + v_2) &= -\frac{LE}{2M}, \\ i\wp'(v_1 - v_2) &= \frac{BD}{2M} \end{aligned} \tag{90}$$

So also Hermite's formula

$$\wp' 2u = \frac{G}{Z^{\frac{3}{2}}}, \tag{91}$$

where G denotes the sextic covariant of Z, leads for the infinite value of z to

$$\frac{\wp' 2v_3}{\sqrt{a}} = \text{coefficient of } 4z \text{ in } Z,$$

or

$$\wp' 2v_3 = 2\sqrt{a}\,\frac{BL}{M^2}. \tag{92}$$

We notice that v_3 is a fraction of the real or the imaginary period, according as a is $+1$ or -1, or as the body is prolate or oblate; and, sometimes, by retaining a, we are able to state the results in a general form, suitable for all cases.

13. The annexed diagrams are intended to illustrate the various cases which may arise, and to exhibit to the eye the separation of the roots of the quartic Z; the curves on the left hand are the typical graphs of the function Z, while the concentric circles on the right show the correspondence of z on the outer circle with the elliptic argument u on the inner circle, the shaded portion representing the limits of the actual variation of z.

Case I applies to the prolate body for which $a = +1$; and all four roots of Z are real, and arranged in the order

$$\infty > z_0 > 1 > z_3 > z > z_2 > -1 > z_1 > -\infty,$$

so that ω_1, ω_3 denoting the real and imaginary periods of $\wp u$, and the letter f being employed to denote generically any fraction,

$$v_1 = f\omega_3,\quad v_2 = \omega_1 + f\omega_3,\quad v_3 = f\omega_1,$$
$$v \text{ or } w = v_1 \pm v_2 = \omega_1 + f\omega_3,$$
$$u = \omega_3 + f\omega_1.$$

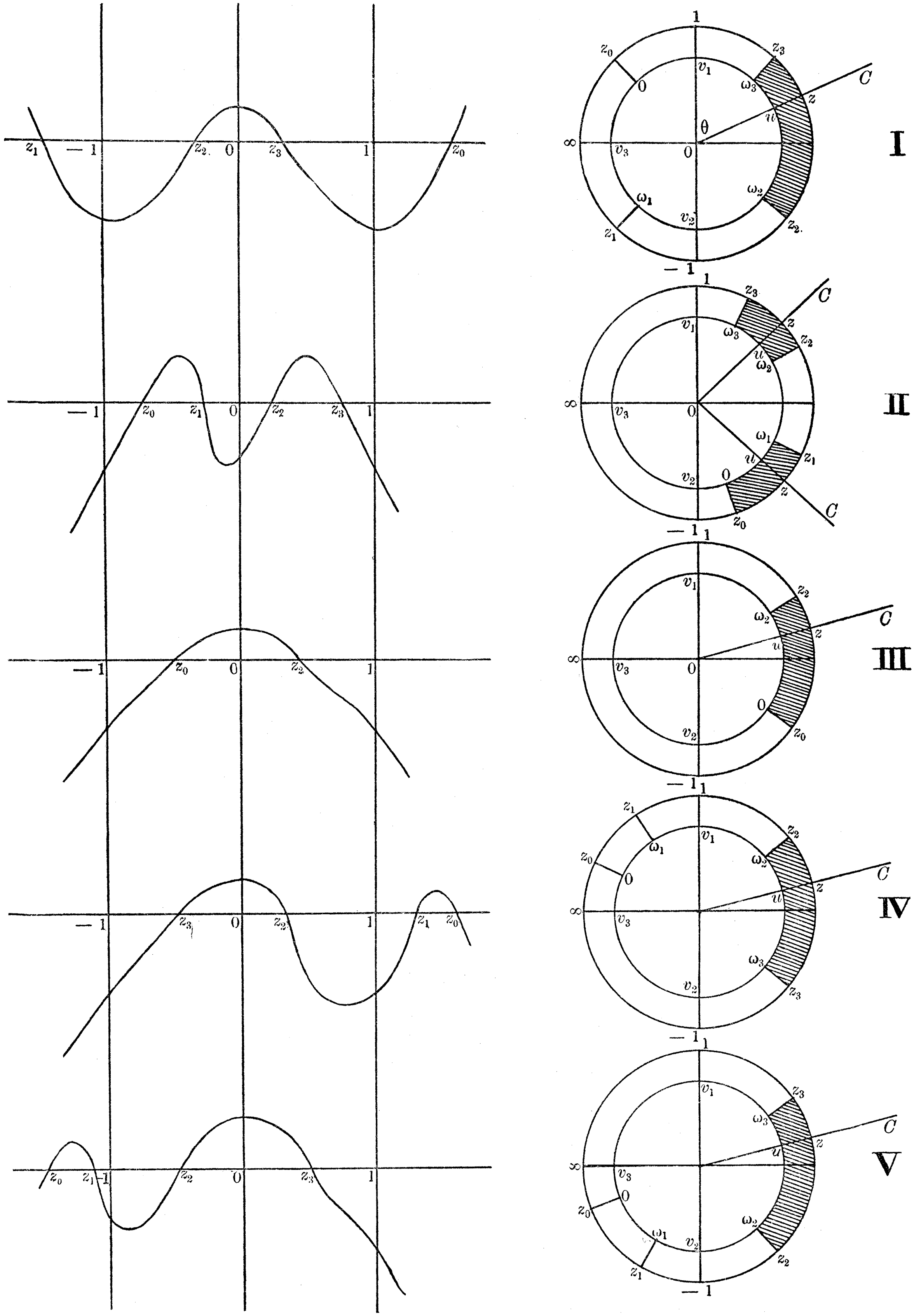
Case II, an oblate body, $a = -1$; all four roots of Z real and arranged in the order

$$\infty > 1 > z_3 > z > z_2 > z_1 > z > z_0 > -1 > -\infty,$$

$$v_1,\ v_2,\ v_3,\ v,\ w = f\omega_3,$$
$$u = \omega_3 + f\omega_1 \text{ or } f\omega_1.$$

Case III, an oblate body, $a = -1$; two real roots of Z and two imaginary:

$$\infty > 1 > z_0 > z > z_2 > -1 > -\infty,$$
$$v_1,\ v_2,\ v_3,\ v,\ w = f\omega_2',$$
$$u = f\omega_2.$$



I
II
III
IV
V

Cases IV and V represent the conditions for an oblate body when the kinetic energy assumes the more general form discussed by Halphen and others, in consequence of which Z takes the more general form, with $a = -1$,

$$Z = -(z^2-1)(z^2-4bz-1+D) - 4\left(\frac{Bx-L}{M}\right)^2$$

or

$$= -(z^2-1)(z^2-4bz-1+E) - 4\left(\frac{Lx-B}{M}\right)^2; \tag{93}$$

and now in

Case IV, $\infty > z_0 > z_1 > 1 > z_2 > z > z_3 > -1 > -\infty$,

$$v_1 = \omega_1 + f\omega_3,\quad v_2 \text{ and } v_3 = f\omega_3,$$
$$v \text{ and } w = \omega_1 + f\omega_3.$$

Case V. $\infty > 1 > z_3 > z > z_2 > -1 > z_1 > z_0 > -\infty$,

$$v_1 \text{ and } v_3 = f\omega_3,\quad v_2 = \omega_1 + f\omega_3,$$
$$v \text{ and } w = \omega_1 + f\omega_3.$$

The term b may arise from the external shape or from circulation in the liquid when the body is perforated or ring-shaped (Basset, Hydrodynamics, I, §193), but the absence of b simplifies considerably the elliptic-function expressions.

14. Putting $z = \pm 1$ in (77) and (79),

$$2\surd(-a)\frac{B-L}{M} = \frac{\wp' v_1 \wp' v_3}{(\wp v_1 - \wp v_3)^2} = \wp(v_1 - v_3) - \wp(v_1 + v_3), \tag{94}$$

$$2\surd(-a)\frac{B+L}{M} = \frac{\wp' v_2 \wp' v_3}{(\wp v_2 - \wp v_3)^2} = \wp(v_2 - v_3) - \wp(v_2 + v_3); \tag{95}$$

$$a = \wp(v_1 - v_3) + \wp(v_1 + v_3) + \wp 2v_3, \tag{96}$$

$$a = \wp(v_2 - v_3) + \wp(v_2 + v_3) + \wp 2v_3. \tag{97}$$

Thence

$$\wp(v_1 - v_3) = \tfrac{1}{2}a - \tfrac{1}{2}\wp 2v_3 + \surd(-a)\frac{B-L}{M}, \tag{98}$$

$$\wp(v_1 + v_3) = \tfrac{1}{2}a - \tfrac{1}{2}\wp 2v_3 - \surd(-a)\frac{B-L}{M}, \tag{99}$$

$$\wp(v_2 - v_3) = \tfrac{1}{2}a - \tfrac{1}{2}\wp 2v_3 + \surd(-a)\frac{B+L}{M}, \tag{100}$$

$$\wp(v_2 + v_3) = \tfrac{1}{2}a - \tfrac{1}{2}\wp 2v_3 - \surd(-a)\frac{B+L}{M}. \tag{101}$$

Again, putting $z = \pm 1$ in (78),

$$\wp'(v_1 - v_3) - \wp' 2v_3 = 2\surd a \{\wp(v_1 - v_3) - \wp 2v_3\}, \tag{102}$$

$$\wp'(v_1 - v_3) = 2\surd a \left\{ \frac{BL}{M^3} + \tfrac{1}{2} a - \tfrac{3}{2}\wp 2v_3 + \surd(-a)\frac{B-L}{M} \right\}$$

$$= 2\surd a \frac{BL}{M^2} - \tfrac{1}{2}\surd aE - 2\surd a \frac{L^2}{M^2} + 2i\frac{B-L}{M}, \tag{103}$$

and

$$\wp'(v_1 - v_3) + \wp'(v_1 + v_3) = 2\surd a \{\wp(v_1 - v_3) - \wp(v_1 + v_3)\}$$

$$= 4i\frac{B-L}{M}, \tag{104}$$

$$\wp'(v_1 + v_3) = -2\surd a \frac{BL}{M^2} + \tfrac{1}{2}\surd aE + 2\surd a \frac{L^2}{M^2} + 2i\frac{B-L}{M}. \tag{105}$$

Similarly

$$\wp'(v_2 - v_3) = \quad 2\surd a \frac{BL}{M^2} + \tfrac{1}{2}\surd aE + 2\surd a \frac{L^2}{M^2} - 2i\frac{B+L}{M}, \tag{106}$$

$$\wp'(v_2 + v_3) = -2\surd a \frac{BL}{M^2} - \tfrac{1}{2}\surd aE - 2\surd a \frac{L^2}{M^2} - 2i\frac{B+L}{M}. \tag{107}$$

Thus

$$\wp(v_1 - v_3) - \wp(v_2 + v_3) = 2\surd(-a)\frac{BL}{M}, \tag{108}$$

$$\wp'(v_1 - v_3) - \wp'(v_2 + v_3) = 4\surd a \frac{BL}{M^2} + 4i\frac{B}{M}; \tag{109}$$

so that

$$\tfrac{1}{2}\frac{\wp'(v_1 - v_3) - \wp'(v_2 + v_3)}{\wp(v_1 - v_3) - \wp(v_2 + v_3)} = -i\frac{L}{M} + \surd a; \tag{110}$$

and squaring,

$$\wp(v_1 + v_2) + \wp(v_1 - v_3) + \wp(v_2 + v_3) = -\frac{L^2}{M^2} - 2\surd(-a)\frac{L}{M} + a. \tag{111}$$

But

$$\wp(v_1 - v_3) + \wp(v_2 + v_3) = a - \wp 2v_3 - 2\surd(-a)\frac{L}{M}; \tag{112}$$

so that

$$\wp(v_1 + v_2) = \wp 2v_3 - \frac{L^2}{M^2}, \tag{113}$$

agreeing with equation (84).

Similarly,

$$\wp(v_1+v_3)-\wp(v_2+v_3)= \quad 2\surd(-a)\frac{L}{M},$$

$$\wp'(v_1+v_3)+\wp'(v_2+v_3)=-4\surd a\frac{BL}{M^2}-4i\frac{L}{M},$$

$$\tfrac{1}{2}\frac{\wp'(v_1+v_3)+\wp'(v_2+v_3)}{\wp(v_1+v_3)-\wp(v_2+v_3)}=+i\frac{B}{M}-\surd a; \tag{114}$$

so that, squaring,

$$\wp(v_1-v_2)+\wp(v_1+v_3)+\wp(v_2+v_3)=-\frac{B^2}{M^2}-2\surd(-a)\frac{B}{M}+a. \tag{115}$$

But

$$\wp(v_1+v_3)+\wp(v_2+v_3)=a-\wp 2v_3-2\surd(-a)\frac{B}{M}, \tag{116}$$

so that

$$\wp(v_1-v_2)=\wp 2v_3-\frac{B^2}{M^2}, \tag{117}$$

agreeing with equation (85).

15. Again, from (110),

$$\tfrac{1}{2}\frac{-\wp'(v_1+v_2)-\wp'(v_1-v_3)}{\wp(v_1+v_2)-\wp(v_1-v_3)}=-i\frac{L}{M}+\surd a, \tag{118}$$

and

$$\wp(v_1+v_2)-\wp(v_1-v_3)=\wp 2v_3-\frac{L^2}{M^2}-\tfrac{1}{2}a+\tfrac{1}{2}\wp 2v_3-\surd(-a)\frac{B-L}{M}$$
$$=\tfrac{1}{4}E-\surd(-a)\frac{B-L}{M}, \tag{119}$$

therefore

$$\wp'(v_1+v_2)=2\left(i\frac{L}{M}-\surd a\right)\{\wp(v_1+v_2)-\wp(v_2-v_3)\}-\wp'(v_2-v_3)$$
$$=\left(i\frac{L}{M}-\surd a\right)\left\{\tfrac{1}{2}E-2\surd(-a)\frac{B-L}{M}\right\}$$
$$-2\surd a\frac{BL}{M^2}+\tfrac{1}{2}\surd aE+2\surd a\frac{L^2}{M^2}-2i\frac{B-L}{M}$$
$$=\tfrac{1}{2}i\frac{LE}{M}, \tag{120}$$

as before in (89).

So also, from (114),

$$\tfrac{1}{2}\frac{\wp'(v_1-v_2)-\wp'(v_2+v_3)}{\wp(v_1-v_2)-\wp(v_2+v_3)}=\tfrac{1}{2}\frac{-\wp'(v_1+v_3)-\wp'(v_2+v_3)}{\wp(v_1+v_3)-\wp(v_2+v_3)}$$
$$=-i\frac{B}{M}+\surd a, \tag{121}$$

and

$$\wp(v_1 - v_2) - \wp(v_2 + v_3) = \wp 2v_3 - \frac{B^2}{M^2} - \tfrac{1}{2}a + \tfrac{1}{2}\wp 2v_3 + \sqrt{(-a)}\,\frac{B+L}{M}$$
$$= \tfrac{1}{4}D + \sqrt{(-a)}\,\frac{B+L}{M}; \qquad (122)$$

therefore

$$\wp'(v_1 - v_2) = \left(-i\frac{B}{M} + \sqrt{a}\right)\left\{\tfrac{1}{2}D + 2\sqrt{(-a)}\,\frac{B+L}{M}\right\}$$
$$- 2\sqrt{a}\,\frac{BL}{M^2} - \tfrac{1}{2}\sqrt{a}D - 2\sqrt{a}\,\frac{B^2}{M^2} - 2i\frac{B+L}{M} = -\tfrac{1}{2}i\frac{BD}{M}, \qquad (123)$$

agreeing with (90).

These verifications are useful in fixing the signs of the various expressions, in the ambiguous cases, of frequent occurrence in these calculations.

16. Returning to (75) and putting in it $u = v_1$, and $z = 1$,

$$\sqrt{a}\,(1 - z_0) = \frac{-\wp' v_3}{\wp v_1 - \wp v_3}, \qquad (124)$$

so that

$$\sqrt{a}\,(1 - z) = \frac{-\wp' v_3(\wp u - \wp v_1)}{(\wp v_1 - \wp v_3)(\wp u - \wp v_3)}; \qquad (125)$$

and similarly,

$$\sqrt{a}\,(-1 - z) = \frac{-\wp' v_3(\wp u - v_2)}{(\wp v_2 - \wp v_3)(\wp u - \wp v_3)}. \qquad (126)$$

Also, from (24), (25), (44), (45) and (77),

$$\frac{G - CrF}{2AnF} = \frac{L - B}{M} = -\frac{1}{2\sqrt{(-a)}}\,\frac{\wp' v_1\,\wp' v_3}{(\wp v_1 - \wp v_3)^2}, \qquad (127)$$

$$\frac{G + CrF}{2AnF} = \frac{L + B}{M} = \frac{1}{2\sqrt{(-a)}}\,\frac{\wp' v_2\,\wp' v_3}{(\wp v_2 - \wp v_3)^2}; \qquad (128)$$

so that in (14)

$$\frac{d\psi_1 i}{du} = i\frac{G - CrF}{2AnF}\,\frac{1}{1 - z}$$
$$= \tfrac{1}{2}\frac{\wp' v_1(\wp u - \wp v_3)}{(\wp v_1 - \wp v_3)(\wp u - \wp v_1)}$$
$$= \tfrac{1}{2}\frac{\wp' v_1}{\wp v_1 - \wp v_3} + \tfrac{1}{2}\frac{\wp' v_1}{\wp u - \wp v_1}$$
$$= \tfrac{1}{2}\zeta(v_1 + v_3) + \tfrac{1}{2}\zeta(v_1 - v_3) - \zeta v_1$$
$$- \tfrac{1}{2}\zeta(u - v_1) + \tfrac{1}{2}\zeta(u + v_1) + \zeta v_1. \qquad (129)$$

Similarly

$$\frac{d\psi_2 i}{du} = i\,\frac{G + CrF}{2AnF}\,\frac{1}{1+z}$$

$$= \tfrac{1}{2}\,\frac{\wp' v_2 (\wp u - \wp v_3)}{(\wp v_2 - \wp v_3)(\wp u - \wp v_2)}$$

$$= \tfrac{1}{2}\,\frac{\wp' v_2}{\wp v_2 - \wp v_3} + \tfrac{1}{2}\,\frac{\wp' v_2}{\wp u - \wp v_2}$$

$$= \tfrac{1}{2}\zeta(v_2 + v_3) + \tfrac{1}{2}\zeta(v_2 - v_3) - \zeta v_2$$

$$- \tfrac{1}{2}\zeta(u - v_2) + \tfrac{1}{2}\zeta(u + v_2) + \zeta v_2. \tag{130}$$

Integrating (129) and (130),

$$\psi_1 i = \tfrac{1}{2}\{\zeta(v_1 + v_3) + \zeta(v_1 - v_3)\}\, nt + \tfrac{1}{2}\log\frac{\sigma(u+v_1)}{\sigma(u-v_1)}, \tag{131}$$

$$\psi_2 i = \tfrac{1}{2}\{\zeta(v_2 + v_3) + \zeta(v_2 - v_3)\}\, nt + \tfrac{1}{2}\log\frac{\sigma(u+v_2)}{\sigma(u-v_2)}; \tag{132}$$

and adding,

$$\psi i = \tfrac{1}{2}\,Qnt + \tfrac{1}{2}\log\frac{\sigma(u+v_1)\,\sigma(u+v_2)}{\sigma(u-v_1)\,\sigma(u-v_2)}, \tag{133}$$

where
$$Q = \zeta(v_1 + v_3) + \zeta(v_1 - v_3) + \zeta(v_2 + v_3) + \zeta(v_2 - v_3). \tag{134}$$

By a theorem of Elliptic Functions,

$$\tfrac{1}{2}\log\frac{\sigma(u+v_1)\,\sigma(u+v_2)}{\sigma(u-v_1)\,\sigma(u-v_2)} = \tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)} + \xi i, \tag{135}$$

where
$$v_1 + v_2 = v, \tag{51}$$

and
$$\xi = \tan^{-1}\frac{(\wp v_1 - \wp v_2)\,\wp' u}{i\wp' v_2(\wp u - \wp v_1) - i\wp' v_1 (\wp u - \wp v_2)}$$

$$= \sin^{-1}\frac{\wp' u}{\sqrt{\{(\wp u - \wp v_1)(\wp u - \wp v_2)(\wp u - \wp v)\}}}, \tag{136}$$

so that we can put

$$\psi i = \tfrac{1}{2}\,Qnt + \xi i + \tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)}, \tag{137}$$

and we have now practically added the parameters v_1 and v_2 of two Elliptic Integrals of the third kind into a single parameter v.

17. We now introduce the Elliptic Integral of the third kind, with elliptic parameter v, in the standard form we shall employ and denote by $I(v)$, namely,

$$I(v) = \tfrac{1}{2}\int \frac{P(v)(s-\sigma) - \surd - \Sigma}{(s-\sigma)\surd S}$$

$$= \tfrac{1}{2} i \log \frac{\sigma(u-v)}{\sigma(u+v)} e^{\left(2\zeta v - i\frac{P}{M}\right)u \text{ (or } nt)}, \tag{138}$$

with

$$\int \frac{ds}{\surd S} = \frac{u}{M} \quad \text{or} \quad \frac{nt}{M}, \tag{139}$$

where $P(v)$ or P is a certain function of v, to be defined hereafter, so chosen as to cancel the secular term when the integral $I(v)$ is *pseudo-elliptic*, in consequence of the parameter v being an aliquot part of a period.

Now from (138),

$$\tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)} = iI(v) - nt\zeta v - \tfrac{1}{2} i \frac{P}{M} nt, \tag{140}$$

and thus

$$\psi i = \tfrac{1}{2}\Big\{\zeta(v_1+v_3) + \zeta(v_2-v_3) - \zeta(v_1+v_2)$$
$$+ \zeta(v_2+v_3) + \zeta(v_1-v_3) - \zeta(v_1+v_2) - i\frac{P}{M}\Big\} nt + iI(v) + \xi i. \tag{141}$$

But, from (99) and (100),

$$\zeta(v_1+v_3) + \zeta(v_2-v_3) - \zeta(v_1+v_2)$$
$$= \surd\{\wp(v_1+v_3) + \wp(v_2-v_3) + \wp(v_1+v_2)\}$$
$$= \surd\Big\{\tfrac{1}{2}a - \tfrac{1}{2}\wp 2v_3 - \surd(-a)\frac{B-L}{M}$$
$$+ \tfrac{1}{2}a - \tfrac{1}{2}\wp 2v_3 + \surd(-a)\frac{B+L}{M} + \wp 2v_3 - \frac{L^2}{M^2}\Big\}$$
$$= \surd\Big\{a + 2\surd(-a)\frac{L}{M} - \frac{L^2}{M^2}\Big\} \quad = i\frac{L}{M} + \surd a, \tag{142}$$

and similarly,

$$\zeta(v_2+v_3) + \zeta(v_1-v_3) - \zeta(v_1+v_2) = i\frac{L}{M} - \surd a. \tag{143}$$

To settle the ambiguity of sign, we may also employ the formulas

$$\zeta(v_1+v_3) + \zeta(v_2-v_3) - \zeta(v_1+v_2) = -\tfrac{1}{2}\frac{\wp'(v_1+v_3) - \wp'(v_2-v_3)}{\wp(v_1+v_3) - \wp(v_2-v_3)},$$

$$\zeta(v_2+v_3) + \zeta(v_1-v_3) - \zeta(v_1+v_2) = -\tfrac{1}{2}\frac{\wp'(v_2+v_3) - \wp'(v_1-v_3)}{\wp(v_2+v_3) - \wp(v_1-v_3)}, \tag{144}$$

and thus

$$\psi i = \left(i\frac{L}{M} - \tfrac{1}{2}i\frac{P}{M}\right)nt + iI(v) + \xi i,$$

and, dropping the factor i,

$$\psi = \frac{L - \frac{1}{2}P(v)}{M}nt + I(v) + \xi, \tag{145}$$

or

$$\psi - pt = I(v) + \xi, \tag{146}$$

where

$$\frac{p}{n} = \frac{L - \frac{1}{2}P(v)}{M}. \tag{147}$$

a different p to the component angular velocity about OA in §1. Without this preliminary determination of pt, the secular term in ψ, the consideration of the pseudo-elliptic solutions would be hopeless.

18. The next chief object of the present paper is the discussion of the Pseudo-Elliptic cases which arise when the parameter v is made an aliquot part, one n^{th}, of a period, so as to be of the form

$$v = \omega_1 + \frac{2\omega_3}{n}, \text{ or } \frac{2\omega_3}{n}, \tag{148}$$

according as the body is prolate or oblate; and when $P(v)$ is at the same time so chosen as to make $I(v)$ an inverse circular function; the preliminary analysis will be found in the paper on "Pseudo-Elliptic Integrals," Proc. London Math. Society, vol. XXV, from which the results required in the sequel will be taken.

In such cases it will be found that we must take

$$\frac{iP(v)}{M} = \zeta v - \frac{\eta v}{\omega}; \tag{149}$$

and

$$iI(v) = \tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)}e^{su}, \tag{150}$$

where

$$v = \frac{2\varpi}{n}, \quad s = \frac{2\eta}{n}; \tag{151}$$

and now $iI(v)$ is the logarithm of a function, analogous to the sn, cn or dn function, which is considered in Halphen's "Fonctions elliptiques," I, p. 224, the function being the n^{th} root of a rational function of $\wp u$ and $\wp' u$.

Further, by taking

$$L = \tfrac{1}{2}P(v), \tag{152}$$

the secular term pt is cancelled in the equations and all trace of elliptic transcendentalism is eliminated, so that the various curves described by points in the axis of the body are, relatively to the moving origin O, purely algebraical curves; these special cases are interesting to discuss and to represent diagrammatically and stereoscopically, like the analogous algebraical Spherical Catenaries and Gyrostat Curves, investigated in the Proc. London Math. Society, vol. XXVII, and drawn by Mr. T. I. Dewar.

19. The curve described by the projection of the moving origin O on a plane perpendicular to Oz is intimately associated with the cone described by the axis of figure OC round Oz; for, denoting the coordinates of the projection of O by α, β, and the advance of O parallel to Oz by γ, then, according to the equations given in Kirchhoff's "Vorlesungen," p. 240,

$$F\alpha = \beta_1 \frac{\partial T}{\partial p} + \beta_2 \frac{\partial T}{\partial q} + \beta_3 \frac{\partial T}{\partial r}, \tag{153}$$

$$F\beta = -\alpha_1 \frac{\partial T}{\partial p} - \alpha_2 \frac{\partial T}{\partial q} - \alpha_3 \frac{\partial T}{\partial r}, \tag{154}$$

$$F\frac{d\gamma}{dt} = u \frac{\partial T}{\partial u} + v \frac{\partial T}{\partial v} + w \frac{\partial T}{\partial w}; \tag{155}$$

where u, v, w again denote the component velocities of O in the directions OA, OB, OC; and α_1, α_2, α_3 denote the cosines of the angles between $O'\alpha$ and OA, OB, OC; and β_1, β_2, β_3 the cosines of the angles between $O'\beta$ and OA, OB, OC; $O'\alpha$, $O'\beta$, $O'\gamma$ being three fixed axes in space, drawn through a fixed origin O', $O'\gamma$ being drawn parallel to Oz.

Expressed by means of Euler's angles θ, ϕ, ψ,

$$\begin{aligned} \alpha_1 &= \cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi, \\ \alpha_2 &= -\cos\theta\sin\phi\cos\psi - \cos\phi\sin\psi, \\ \alpha_3 &= \sin\theta\cos\psi; \end{aligned} \tag{156}$$

$$\begin{aligned} \beta_1 &= \cos\theta\cos\phi\sin\psi + \sin\phi\cos\psi, \\ \beta_2 &= -\cos\theta\sin\phi\sin\psi + \cos\phi\cos\psi, \\ \beta_3 &= \sin\theta\sin\psi; \end{aligned} \tag{157}$$

while

$$\begin{aligned} p &= \sin\phi\dot\theta - \sin\theta\cos\phi\dot\psi, \\ q &= \cos\phi\dot\theta + \sin\theta\sin\phi\dot\psi, \\ r &= \dot\phi + \cos\theta\dot\psi; \end{aligned} \tag{158}$$

so that, after reduction,

$$F\alpha = A\cos\psi\dot{\theta} + (Cr - A\cos\theta\dot{\psi})\sin\theta\sin\psi.$$

$$F\beta = A\sin\psi\dot{\theta} - (Cr - A\cos\theta\dot{\psi})\sin\theta\cos\psi, \tag{159}$$

$$F\frac{d\alpha}{dt} = aAn^2\sin\theta\cos\theta\cos\psi,$$

$$F\frac{d\beta}{dt} = aAn^2\sin\theta\cos\theta\sin\psi,$$

$$F\frac{d\gamma}{dt} = \frac{F^2}{P} + aAn^2\cos^2\theta. \tag{160}$$

20. Changing to polar coordinates ρ, ϖ in the plane $O'\alpha\beta$, so that

$$\alpha = \rho\cos\varpi,\quad \beta = \rho\sin\varpi, \tag{161}$$

then

$$F(\alpha + \beta i) = F\rho e^{\varpi i} = \{A\dot{\theta} - i\sin\theta(Cr - A\cos\theta\dot{\psi})\}e^{\psi i}, \tag{162}$$

and

$$F\rho\cos(\psi - \varpi) = A\dot{\theta} = -\frac{An\sqrt{Z}}{\sin\theta}, \tag{163}$$

$$F\rho\sin(\psi - \varpi) = (Cr - A\cos\theta\dot{\psi})\sin\theta$$
$$= \frac{CrF - G\cos\theta}{F\sin\theta}. \tag{164}$$

Thus, squaring and adding,

$$F^2\rho^2 = \frac{A^2n^2F^2Z + (CrF - Gz)^2}{F^2(1 - z^2)}$$
$$= A^2n^2a(aE + 1 - z^2); \tag{165}$$

and, dividing,

$$\tan(\psi - \varpi) = -\frac{CrF - Gz}{AnF\sqrt{Z}}, \tag{166}$$

$$\varpi = \psi + \tan^{-1}\frac{CrF - Gz}{AnF\sqrt{Z}}$$
$$= \psi + \sin^{-1}\frac{CrF - G\cos\theta}{F^2\rho\sin\theta}$$
$$= \psi + \cos^{-1}\frac{\sqrt{Z}}{\sqrt{\{a(1 - z^2)(aE + 1 - z^2)\}}}; \tag{167}$$

so that, from equation (146),

$$\varpi = pt + I(v) + \xi + \sin^{-1}\frac{CrF - G\cos\theta}{F^2\rho\sin\theta}, \tag{168}$$

and ϖ and ψ depend upon the same elliptic integral $I(v)$.

We find in fact, by differentiation of (167),

$$\frac{d\varpi}{dz} = \frac{d\psi}{dz} + \frac{d}{dz}\sin^{-1}\frac{CrF - Gz}{AnF\surd\{a(1-z^2)(aE+1-z^2)\}}$$

$$= \frac{G - CrFz}{AnF(1-z^2)}\frac{1}{\surd Z}$$

$$+ \frac{\dfrac{CrF - Gz}{AnF\surd\{a(1-z^2)(aE+1-z^2)\}}\left(\dfrac{-G}{CrF - Gz} + \dfrac{z}{1-z^2} + \dfrac{z}{aE+1-z^2}\right)}{\dfrac{\surd Z}{\surd\{a(1-z^2)(aE+1-z^2)\}}}$$

$$= \left\{\frac{G - CrFz}{AnF(1-z^2)} - \frac{G}{AnF} + \frac{CrFz - Gz^3}{AnF(1-z^2)} + \frac{CrFz - Gz^2}{AnF(aE+1-z^2)}\right\}\frac{1}{\surd Z}$$

$$= \frac{CrFz - Gz^2}{AnF(aE+1-z^2)}\frac{1}{\surd Z}$$

$$= 2\frac{Bz - Lz^2}{M(aE+1-z^2)}\frac{1}{\surd Z}. \tag{169}$$

Now if w_1, w_2 denote the values of the elliptic argument u corresponding to

$$aE + 1 - z^2 = 0, \tag{170}$$

it follows by Abel's theorem and the theory of elliptic functions that

$$w_1 + w_2 = v_1 + v_2 = v. \tag{171}$$

So also if t_1, t_2 denote the values of u corresponding to

$$aD + 1 - z^2 = 0, \tag{172}$$

then

$$t_1 - t_2 = v_1 - v_2 = w. \tag{173}$$

21. From equations (125) and (126),

$$\sin^2\theta = 1 - z^2 = -a\frac{\wp'^2 v_3(\wp u - \wp v_1)(\wp u - \wp v_2)}{(\wp v_1 - \wp v_3)(\wp v_2 - \wp v_3)(\wp u - \wp v_3)^2}$$

$$= C\frac{\sigma(u - v_1)\,\sigma(u + v_1)\,\sigma(u - v_2)\,\sigma(u + v_2)}{\sigma^2(u - v_3)\,\sigma^2(u + v_3)}, \tag{174}$$

and from (133),

$$e^{2\psi i} = \frac{\sigma(u + v_1)\,\sigma(u + v_2)}{\sigma(u - v_1)\,\sigma(u - v_2)}e^{Qnt}; \tag{175}$$

so that by multiplication,

$$\sin\theta e^{\psi i} = \surd C\frac{\sigma(u + v_1)\,\sigma(u + v_2)}{\sigma(u - v_3)\,\sigma(u + v_3)}e^{\frac{1}{2}Qnt}. \tag{176}$$

Or, in Klein's manner, with the stereographic projection,

$$\tan \tfrac{1}{2}\theta e^{\psi i} = C' \frac{\sigma(u+v_1)\,\sigma(u+v_2)}{\sigma(u-v_1)\,\sigma(u-v_2)} e^{\frac{1}{2}Qnt}. \tag{177}$$

Similarly we find

$$\rho e^{\varpi i} = C'' \frac{\sigma(u+w_1)\,\sigma(u+w_2)}{\sigma(u-v_3)\,\sigma(u+v_3)} e^{\frac{1}{2}Qnt}, \tag{178}$$

so that the expressions in Halphen's equations (52), F. E. II, p. 164, appear to require correction; however, the curve of (α, β) is intimately connected with the cone of (θ, ψ), and the two are pseudo-elliptic, and even algebraical, at the same time.

As for u, v and p, q, they depend upon the same elliptic functions as Euler's angle ϕ; for

$$P(u+vi) = -F\sin\theta\, e^{-\phi i}, \tag{179}$$

$$p + qi = (-\sin\theta\dot{\psi} + i\dot{\theta})\, e^{-\phi i}$$

$$= -n \frac{2\dfrac{L - Bz}{M} + i\sqrt{Z}}{\sqrt{(1-z^2)}} e^{-\phi i}, \tag{180}$$

which are pseudo-elliptic together with ϕ; and this is the case when

$$v_1 - v_2 = t_1 - t_2 = w \tag{181}$$

is an aliquot part of a period; but these cases are not so interesting from the dynamical point of view.

22. In the pseudo-elliptic case of the motion of a prolate solid, when the parameter

$$v = \omega_1 + \frac{2\omega_3}{n}, \tag{182}$$

where n is an odd integer, the expressions for ψ and ϖ must be of the form

$$\psi - pt = \frac{1}{n}\cos^{-1}\frac{Hz^{n-1} + H_1 z^{n-2} + \ldots + H_{n-1}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z_0 - z\,.\,z - z_1)}$$

$$= \frac{1}{n}\sin^{-1}\frac{Kz^{n-1} + K_1 z^{n-2} + \ldots + K_{n-1}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z_3 - z\,.\,z - z_2)}, \tag{183}$$

and

$$\varpi - pt = \frac{1}{n}\cos^{-1}\frac{Iz^{n-1} + I_1 z^{n-2} + \ldots + I_{n-1}}{(E+1-z^2)^{\frac{1}{2}n}}\sqrt{(z_0 - z\,.\,z - z_1)}$$

$$= \frac{1}{n}\sin^{-1}\frac{Jz^{n-1} + J_1 z^{n-2} + \ldots + J_{n-1}}{(E+1-z^2)^{\frac{1}{2}n}}\sqrt{(z_3 - z\,.\,z - z_2)}. \tag{184}$$

But with a parameter

$$v = \omega_1 + \frac{\omega_2}{n}, \tag{185}$$

where n is an integer, the z_1 and z_2 or z_3 must change places, as well as the $\cos^{-1}$ and $\sin^{-1}$, taking $\frac{dz}{dt}$ as positive at the start.

With an oblate body the parameter v is a fraction of the imaginary period, and when

$$v = \frac{\omega_3}{n}, \tag{186}$$

where n is an integer, we must have

$$\begin{aligned}\psi - pt &= \frac{1}{n}\cos^{-1}\frac{Hz^{n-1} + H_1 z^{n-2} + \ldots + H_{n-1}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z \sim z_1 . z \sim z_2)}\\ &= \frac{1}{n}\sin^{-1}\frac{Kz^{n-1} + K_1 z^{n-2} + \ldots + K_{n-1}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z_3 - z . z - z_0)},\end{aligned} \tag{187}$$

with a similar expression for $\varpi - pt$.

When, in the motion of an oblate body, the parameter

$$v = \frac{2\omega_3}{n}, \tag{188}$$

where n is an odd integer, the quartic Z will not be divided into quadratic factors, and we must have

$$\begin{aligned}\psi - pt &= \frac{1}{n}\cos^{-1}\frac{Hz^{n} + H_1 z^{n-1} + \ldots + H_{n}}{(1-z^2)^{\frac{1}{2}n}}\\ &= \frac{1}{n}\sin^{-1}\frac{Kz^{n-2} + K_1 z^{n-3} + \ldots + K_{n-2}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{Z},\end{aligned} \tag{189}$$

with a similar expression for $\varpi - pt$.

23. Having now chosen a simple numerical value of n, for a case worked out in the paper on "Pseudo-Elliptic Integrals," and having expressed the s_1, s_2, s_3 in (50), or Halphen's x and y in (49), in terms of a single parameter c, as well as σ, $\sqrt{-\Sigma}$ and $P(v)$, and having chosen an arbitrary value of L, the quickest practical procedure for the determination of the coefficients H, K and

I, J, appears to be effected from the identical equations obtained by squaring and adding, and also from the differentiation of the two equivalent forms for ψ and ϖ.

Thus in consequence of the relations

$$\frac{dz}{dt} = n\sqrt{Z}, \tag{19}$$

$$\frac{d\psi}{dt} = 2\,\frac{G - CrFz}{AF(1-z^2)}, \tag{12}$$

we have

$$\frac{d\psi}{dz} = 2\,\frac{L - Bz}{M(1-z^2)}\,\frac{1}{\sqrt{Z}}; \tag{46}$$

and putting

$$\psi - pt = \chi, \tag{190}$$

$$\frac{d\chi}{dz} = 2\,\frac{L - Bz}{M(1-z^2)}\,\frac{1}{\sqrt{Z}} - \frac{p}{n}\,\frac{1}{\sqrt{Z}},$$

$$(1-z^2)\sqrt{Z}\,\frac{d\chi}{dz} = 2\,\frac{L - Bz}{M} - \frac{p}{n}(1-z^2)$$

$$= \frac{L - \frac{1}{2}P(v)}{M}\,z^2 - 2\,\frac{B}{M}\,z + \frac{L + \frac{1}{2}P(v)}{M}. \tag{191}$$

So also

$$(aE + 1 - z^2)\sqrt{Z}\,\frac{d(\varpi - pt)}{dz}$$

$$= -\frac{L + \frac{1}{2}P}{M}\,z^2 + 2\,\frac{B}{M}\,z - \frac{L - \frac{1}{2}P}{M}\,(aE + 1). \tag{192}$$

24. Begin with the simplest pseudo-elliptic case of all, in which

$$v = \omega_a \text{ and } \wp' v = 0; \tag{193}$$

so that, from (72),

$$L = 0, \tag{194}$$

or

$$E = 0. \tag{195}$$

When $L = 0$ or $G = 0$,

$$\frac{d\psi}{dz} = -2\,\frac{B}{M}\,\frac{z}{(1-z^2)\sqrt{Z}}, \tag{196}$$

$$\frac{d\varpi}{dz} = 2\,\frac{B}{M}\,\frac{z}{(aE + 1 - z^2)\sqrt{Z}}, \tag{197}$$

where Z is a quadratic in z^2, so that taking z^2 as independent variable, ψ and ϖ are non-elliptic, and integrating,

$$\psi = \tfrac{1}{2}\sin^{-1}\frac{2\dfrac{B}{M}\surd Z}{\Delta(1-z^2)}, \tag{198}$$

$$\varpi = \tfrac{1}{2}\sin^{-1}\frac{2\dfrac{B}{M}\surd Z}{\Delta(aE+1-z^2)}, \tag{199}$$

where Δ denotes the discriminant of Z.

For a prolate body, $a = +1$,

$$Z = (z_1^2 - z^2)(z_2^2 - z^2), \tag{200}$$

and

$$\psi = \cos^{-1}\sqrt{\frac{(1-z_2^2)(z_1^2-z^2)}{(z_1^2-z_2^2)(1-z^2)}},$$

$$= \sin^{-1}\sqrt{\frac{(z_1^2-1)(z_2^2-z^2)}{(z_1-z_2^2)(1-z^2)}}, \tag{201}$$

$$\varpi = \sin^{-1}\sqrt{\frac{(E+1-z_2^2)(z_1^2-z^2)}{(z_1^2-z_2^2)(E+1-z^2)}}$$

$$= \cos^{-1}\sqrt{\frac{(z_1^2-1-E)(z_2^2-z^2)}{(z_1^2-z_2^2)(E+1-z^2)}}; \tag{202}$$

which may be written

$$\surd(z_1^2-z_2^2)\sin\theta e^{\psi i} = \surd(1-z_2^2)\surd(z_1^2-z^2) + i\surd(z_1^2-1)\surd(z_2^2-z^2), \tag{203}$$

$$\surd(z_1^2-z_2^2)\frac{F}{An}\rho e^{\varpi i} = \surd(z_1^2-1-E)\surd(z_2^2-z^2) + i\surd(E+1-z_2^2)\surd(z_1^2-z^2), \tag{204}$$

and these equations prove that the cone described by the axis OC round the fixed direction Oz is a quadric cone, while the curve of (α, β) is a conic section, an ellipse.

For an oblate body, $a = -1$; and in Case II,

$$Z = (z_0^2 - z^2)(z^2 - z_1^2) \tag{205}$$

and

$$\psi = \cos^{-1}\sqrt{\frac{(1-z_0^2)(z^2-z_1^2)}{(z_0^2-z_1^2)(1-z^2)}}$$

$$= \sin^{-1}\sqrt{\frac{(1-z_1^2)(z_0^2-z^2)}{(z_0^2-z_1^2)(1-z^2)}}, \tag{206}$$

$$\varpi = \cos^{-1}\sqrt{\frac{(z_0^2 - 1 + E)(z^2 - z_1^2)}{(z_0^2 - z_1^2)(z^2 - 1 + E)}}$$

$$= \sin^{-1}\sqrt{\frac{(z_1^2 - 1 + E)(z_0^2 - z^2)}{(z_0^2 - z_1^2)(z^2 - 1 + E)}}; \tag{207}$$

or

$$\sqrt{(z_0^2 - z_1^2)}\sin\theta e^{\psi i} = \sqrt{(1 - z_0^2)}\sqrt{(z^2 - z_1^2)} + i\sqrt{(1 - z_1^2)}\sqrt{(z_0^2 - z^2)}, \tag{208}$$

$$\sqrt{(z_0^2 - z_1^2)}\frac{F}{An}\rho e^{\varpi i} = \sqrt{(z_0^2 - 1 + E)}\sqrt{(z^2 - z_1^2)} + i\sqrt{(z_1^2 - 1 + E)}\sqrt{(z_0^2 - z^2)}, \tag{209}$$

so that OC describes a quadric cone, and (α, β) describes an ellipse, as before.

In case III, with $L = 0$, we must put

$$Z = (z_1^2 + z^2)(z_2^2 - z^2), \tag{210}$$

and now

$$\sqrt{(z_1^2 + z_2^2)}\sin\theta e^{\psi i} = \sqrt{(1 - z_2^2)}\sqrt{(z_1^2 + z^2)} + i\sqrt{(1 + z_1^2)}\sqrt{(z_2^2 - z^2)}, \tag{211}$$

$$\sqrt{(z_1^2 + z_2^2)}\frac{F}{An}\rho e^{\varpi i} = \sqrt{(z_2^2 - 1 + E)}\sqrt{(z_1^2 + z^2)} + i\sqrt{(-z_1^2 - 1 + E)}\sqrt{(z_2^2 - z^2)}. \tag{212}$$

The curve of (α, β) is still a conic section, and OC describes a quadric cone relatively to Oz, but the azimuth ψ oscillates between

$$\pm\cos^{-1}\sqrt{\frac{1 - z_2^2}{z_1^2 + z_2^2}z_1^2} = \pm\sin^{-1}\sqrt{\frac{1 + z_1^2}{z_1^2 + z_2^2}z_2^2}, \tag{213}$$

obtained by putting $z = 0$.

Next, with $E = 0$, we have

$$Z = a(z^2 - 1)^2 - 4\left(\frac{Lz - B}{M}\right)^2. \tag{214}$$

With an oblate body, $a = -1$, and Z is always negative, so that no solution exists.

But with a prolate body and $a = +1$,

$$Z = \left(1 - 2\frac{B}{M} + 2\frac{Lz}{M} - z^2\right)\left(1 + 2\frac{B}{M} - 2\frac{Lz}{M} - z^2\right), \tag{215}$$

and

$$\psi - pt = \sin^{-1} \frac{\sqrt{\left(1 - 2\dfrac{B}{M} + 2\dfrac{Lz}{M} - z^2\right)}}{\sqrt{2}\sqrt{(1-z^2)}}$$

$$= \cos^{-1} \frac{\sqrt{\left(1 + 2\dfrac{B}{M} - 2\dfrac{Lz}{M} - z^2\right)}}{\sqrt{2}\sqrt{(1-z^2)}}$$

$$= \tfrac{1}{2} \sin^{-1} \frac{\sqrt{Z}}{1-z^2}, \tag{216}$$

where

$$\frac{p}{n} = \frac{L}{M}, \text{ and } \frac{dz}{dt} = n\sqrt{Z}, \tag{216a}$$

as may be verified by differentiation.

Also, with $E = 0$,

$$\frac{d\varpi}{dz} = 2\frac{Bz - Lz^2}{M(1-z^2)} \frac{1}{\sqrt{Z}}, \tag{217}$$

so that

$$\frac{d\varpi}{dz} + \frac{d\psi}{dz} = 2\frac{L}{M} \frac{1}{\sqrt{Z}} = 2p\frac{dt}{dz}, \tag{218}$$

$$\varpi + \psi = 2pt + \frac{\pi}{2}; \tag{219}$$

and thus

$$\varpi - pt = \cos^{-1} \frac{\sqrt{\left(1 - 2\dfrac{B}{M} + 2\dfrac{Lz}{M} - z^2\right)}}{\sqrt{2}\sqrt{(1-z^2)}}$$

$$= \sin^{-1} \frac{\sqrt{\left(1 + 2\dfrac{B}{M} - 2\dfrac{Lz}{M} - z^2\right)}}{\sqrt{2}\sqrt{(1-z^2)}}$$

$$= \tfrac{1}{2} \sin^{-1} \frac{\sqrt{Z}}{1-z^2}. \tag{220}$$

25. With $r = 0$ or $B = 0$, the case of no rotation round OC,

$$w = \omega_a, \tag{221}$$

and

$$Z = a(z^2 - 1)(z^2 - 1 - aD) - 4\frac{L^2}{M^2}$$

$$= a(z^2 - 1)(z^2 - 1 - aE) - 4\frac{L^2}{M^2} z^2, \tag{222}$$

and now equations (19) and (12) can be integrated by means of the ordinary Jacobian elliptic functions of the second stage.

When $B = 0$, then either N_1 or $N_3 = 0$, according as the body is prolate or oblate; so that

$$L = \sqrt{(s_1 - \sigma)}, \text{ or } -\sqrt{(s_3 - \sigma)},$$

and

$$M = \sqrt{(\sigma - s_3)} - \sqrt{(\sigma - s_2)}, \text{ or } \sqrt{(s_1 - \sigma)} + \sqrt{(s_2 - \sigma)};$$

$$1 + aE = \left\{\frac{\sqrt{(\sigma - s_3)} + \sqrt{(\sigma - s_2)}}{\sqrt{(\sigma - s_3)} - \sqrt{(\sigma - s_2)}}\right\}^2 \text{ or } \left\{\frac{\sqrt{(s_1 - \sigma)} - \sqrt{(s_2 - \sigma)}}{\sqrt{(s_1 + \sigma)} + \sqrt{(s_2 - \sigma)}}\right\}^2;$$

and

$$Z = \left[z^2 - \left\{\frac{\sqrt{(s_1 - s_3)} - \sqrt{(s_1 - s_2)}}{\sqrt{(\sigma - s_3)} - \sqrt{(\sigma - s_2)}}\right\}^2\right]\left[z^2 - \left\{\frac{\sqrt{(s_1 - s_3)} + \sqrt{(s_1 - s_2)}}{\sqrt{(\sigma - s_3)} - \sqrt{(\sigma - s_2)}}\right\}^2\right]$$

or

$$-\left[z^2 - \left\{\frac{\sqrt{(s_1 - s_3)} - \sqrt{(s_2 - s_3)}}{\sqrt{(s_1 - \sigma)} + \sqrt{(s_2 - \sigma)}}\right\}^2\right]\left[z^2 - \left\{\frac{\sqrt{(s_1 - s_3)} + \sqrt{(s_2 - s_3)}}{\sqrt{(s_1 - \sigma)} + \sqrt{(s_2 - \sigma)}}\right\}^2\right].$$

With an oblate body we shall find that this makes $H = 0$, $K = 1$, $K_1 = 0, \ldots.,$ while in the case of a prolate body the parameter must be of the form (182) for similar reductions to take place; and for the parameter in (185) we shall have $H = K = \frac{1}{2}\sqrt{2}$, when B and $N_1 = 0$.

26. Passing on to the next simplest pseudo-elliptic case of bisection of a period, by taking a parameter

$$v = \omega_1 + \tfrac{1}{2}\omega_3 \tag{223}$$

for a prolate body, and writing χ for $\psi - pt$ throughout, the result must be of the form

$$\chi = \tfrac{1}{2}\cos^{-1}\frac{Hz + H_1}{1 - z^2}\sqrt{(z_0 - z \,.\, z_3 - z)}$$

$$= \tfrac{1}{2}\sin^{-1}\frac{Kz + K_1}{1 - z^2}\sqrt{(z - z_2 \,.\, z - z_1)} \tag{224}$$

with

$$K^2 + H^2 = 1. \tag{225}$$

The solution must be built up from the associated pseudo-elliptic integral

$$I(\omega_1 + \tfrac{1}{2}\omega_3) = \tfrac{1}{2}\int \frac{(c + c^2 - s) - 2(c + c^2)}{(c + c^2 - s)\sqrt{S}}\, ds$$

$$= \tfrac{1}{2}\sin^{-1}\frac{\sqrt{\{(1+c)^2 - s\,.\,c^2 - s\}}}{c + c^2 - s}$$

$$= \tfrac{1}{2}\cos^{-1}\frac{\sqrt{s}}{c + c^2 - s}\,; \tag{226}$$

in which

$$s_1 = (1 + c)^2, \quad s_2 = c^2, \qquad s_3 = 0; \tag{227}$$

$$P(v) = 1, \qquad \sigma = c + c^2, \quad \sqrt{-\Sigma} = 2(c + c^2). \tag{228}$$

Then, with the preceding notation of (54),

$$N_1^2 = L^2 - 1 - c,$$
$$N_2^2 = L^2 + 0 + c,$$
$$N_3^2 = L^2 + 0 + c + c^2, \tag{229}$$

so that

$$N_3^2 + N_1^2 N_2^2 = L^4, \tag{230}$$
$$N_3^2 - L^2 \quad = c + c^2, \tag{231}$$
$$N_1^2 + N_2^2 \quad = 2L^2 - 1; \tag{232}$$

and from (59) and (63)

$$\frac{B}{M} = \frac{N_1 N_2 N_3}{LM^2}, \tag{233}$$

$$M^2 = L^2 - 1 + c + c^2 - \frac{2c + 2c^2}{L},$$

$$LM^2 = 2L^2 - L + (L - 2)N_3^2,$$

$$Mz_0 = -N_1 + N_2 + N_3,$$
$$Mz_3 = +N_1 - N_2 + N_3,$$
$$Mz_2 = +N_1 + N_2 - N_3,$$
$$Mz_1 = -N_1 - N_2 - N_3, \tag{235}$$

arranged in the order

$$\infty > z_0 > 1 > z_3 > z > z_2 > -1 > z_1 > -\infty;$$

with $\qquad N_1 < N_2 < N_3.$

To determine the coefficients H, H_1, K, K_1 in equations (224), differentiate with respect to z; then since

$$(z_0 - z)(z_3 - z) = z^2 - 2\frac{N_3}{M}z + z_0 z_3,$$

$$(z - z_2)(z - z_1) = z^2 + 2\frac{N_3}{M}z + z_1 z_2, \tag{236}$$

$$\frac{d\chi}{dz} = \frac{-\frac{1}{2}\dfrac{Hz+H_1}{1-z^2}\sqrt{\left(z^2-2\dfrac{N_3}{M}z+z_0z_3\right)}\left(\dfrac{H}{Hz+H_1}+\dfrac{z-\dfrac{N_3}{M}}{z^2-2\dfrac{N_3}{M}z+z_0z_3}+\dfrac{2z}{1-z^2}\right)}{\left(\dfrac{Kz+K_1}{1-z^2}\right)\sqrt{\left(z^2+2\dfrac{N_3}{M}z+z_1z_2\right)}}$$

$$= \frac{\frac{1}{2}\dfrac{Kz+K_1}{1-z^2}\sqrt{\left(z^2+2\dfrac{N_3}{M}z+z_1z_2\right)}\left(\dfrac{K}{Kz+K_1}+\dfrac{z+\dfrac{N_3}{M}}{z^2+2\dfrac{N_3}{M}z+z_1z_2}+\dfrac{2z}{1-z^2}\right)}{\dfrac{Hz+H_1}{1-z^2}\sqrt{\left(z^2-2\dfrac{N_3}{M}z+z_0z_3\right)}}$$

$$= -\tfrac{1}{2}\,\frac{H\left(z^2-2\dfrac{N_3}{M}z+z_0z_3\right)\left(1-z^2\right)+\left(z-\dfrac{N_3}{M}\right)(Hz+H_1)(1-z^2)+2z\,(Hz+H_1)\left(z^2-2\dfrac{N_3}{M}z+z_0z_3\right)}{(Kz+K_1)(1-z^2)\sqrt{Z}}$$

$$= \tfrac{1}{2}\,\frac{K\left(z^2+2\dfrac{N_3}{M}z+z_1z_2\right)(1-z^2)+\left(z+\dfrac{N_3}{M}\right)(Kz+K_1)(1-z^2)+2z\,(Kz+K_1)\left(z^2+2\dfrac{N_3}{M}z+z_1z_2\right)}{(Hz+H_1)(1-z^2)\sqrt{Z}}, \quad (237)$$

each of which can be equated to (191), so that

$$(Kz+K_1)\left(\frac{2L-P}{M}z^2-4\frac{B}{M}z+\frac{2L+P}{M}\right),$$

$$= H\left(z^2-2\frac{N_3}{M}z+z_0z_3\right)(z^2-1)+\left(z-\frac{N_3}{M}\right)(Hz+H_1)(z^2-1)-2\,(Hz^2+H_1z)\left(z^2-2\frac{N_3}{M}z+z_0z_3\right)$$

$$= O\,.\,z^4+\left(-H_1+\frac{N_3}{M}H\right)z^3+\left\{-(2+z_0z_3)\,H+3\,\frac{N_3}{M}H_1\right\}z^2+\ldots., \quad (238a)$$

$$(Hz+H_1)\left(\frac{2L-P}{M}z^2-4\frac{B}{M}z+\frac{2L+P}{M}\right)$$

$$= -K\left(z^2+2\frac{N_3}{M}z+z_1z_2\right)(z^2-1)-\left(z+\frac{N_3}{M}\right)(Kz+K_1)(z^2-1)+2\,(Kz^2+K_1)\left(z^2+2\frac{N_3}{M}z+z_1z_2\right)$$

$$= 0\,.\,z^4+\left(-K_1-\frac{N_3}{M}K\right)z^3+\left\{(2+z_1z_2)\,K+3\,\frac{N_3}{M}K_1\right\}z^2+\ldots.; \quad (238b)$$

and equating the coefficients in these identities gives us equations enough, and to spare, to determine H, H_1, K, K_1.

Putting $z = \pm 1$, gives the relations

$$2(K + K_1)\frac{L - B}{M} = -(H + H_1)\left(1 - 2\frac{N_3}{M} + z_0 z_3\right),$$

$$2(H + H_1)\frac{L - B}{M} = \quad (K + K_1)\left(1 + 2\frac{N_3}{M} + z_1 z_2\right),$$

$$2(K - K_1)\frac{L + B}{M} = \quad (H - H_1)\left(1 + 2\frac{N_3}{M} + z_0 z_3\right),$$

$$2(H - H_1)\frac{L + B}{M} = -(K - K_1)\left(1 - 2\frac{N_3}{M} + z_1 z_2\right), \tag{239}$$

which are useful forms to serve for verification.

Working in this manner we find after considerable reduction that

$$\frac{H}{K} = \frac{L^2 - N_1 N_2}{N_3}, \quad \frac{K}{H} = \frac{L^2 + N_1 N_2}{N_3}, \tag{240}$$

and, squaring and adding in (224),

$$K^2 + H^2 = 1, \tag{241}$$

so that

$$\frac{1}{HK} = \frac{K}{H} + \frac{H}{K} = \frac{2L^2}{N_3},$$

or

$$2HK = \frac{N_3}{L^2}, \tag{242}$$

and

$$\frac{K}{H} - \frac{H}{K} = 2\frac{N_1 N_2}{N_3},$$

so that

$$K^2 - H^2 = \frac{N_1 N_2}{L^2}; \tag{243}$$

$$K^2 = \frac{L^2 + N_1 N_2}{2L^2},$$

$$H^2 = \frac{L^2 - N_1 N_2}{2L^2}. \tag{244}$$

Also

$$\frac{H_1}{K} = \quad \frac{(L-1)^2 - N_1 N_2}{M},$$

$$\frac{K_1}{H} = \frac{-(L-1)^2 - N_1 N_2}{M}, \tag{245}$$

so that

$$\frac{H_1}{K} + \frac{K_1}{H} = -2\frac{N_1 N_2}{M}, \tag{246}$$

as is verified by squaring and adding; and all other relations are also found to verify.

27. In the case of the oblate body with parameter

$$v = \tfrac{1}{2}\omega_3, \tag{247}$$

we must take

$$\begin{aligned}\chi &= \tfrac{1}{2}\cos^{-1}\frac{Hz + H_1}{1 - z^2}\sqrt{(z - z_1 . z - z_2)} \\ &= \tfrac{1}{2}\sin^{-1}\frac{Kz + K_1}{1 - z^2}\sqrt{(z_3 - z . z - z_0)},\end{aligned} \tag{248}$$

and build up upon the pseudo-elliptic integral

$$\begin{aligned}I(\tfrac{1}{2}\omega_3) &= \tfrac{1}{2}\int\frac{(1 + 2c)(s + c + c^2) - 2(1 + 2c)(c + c^2)}{(s + c + c^2)\sqrt{S}}\,ds \\ &= \tfrac{1}{2}\sin^{-1}\frac{\sqrt{\{s - (1 + c)^2 . s - c^2\}}}{s + c + c^2} \\ &= \tfrac{1}{2}\cos^{-1}\frac{(1 + 2c)\sqrt{s}}{s + c + c^2},\end{aligned} \tag{249}$$

so that, with the same s_1, s_2, s_3, we take

$$P(v) \text{ or } P = 1 + 2c, \quad \sigma = -c - c^2, \quad \sqrt{-\Sigma} = 2(1 + 2c)(c + c^2); \tag{250}$$

and now, with (52),

$$\begin{aligned}N_a^2 &= s_a - \sigma - L^2, \\ N_1^2 &= 1 + 3c + 2c^2 - L^2, \\ N_2^2 &= c + 2c^2 - L^2, \\ N_3^2 &= c + c^2 - L^2,\end{aligned} \tag{251}$$

so that

$$N_1^2 + N_2^2 = P^2 - 2L^2, \tag{252}$$

$$N_3^2 + L^2 = c + c^2, \tag{253}$$

$$N_1^2 N_2^2 - L^4 = P^2 N_3^2; \tag{254}$$

and from (59) and (63),

$$\frac{B}{M} = -\frac{N_1 N_2 N_3}{L M^2}, \tag{255}$$

$$\begin{aligned}M^2 &= 1 + 5c + 5c^2 - L^2 - \frac{2(1 + 2c)(c + c^2)}{L} \\ &= P^2 + N_3^2 - 2\frac{P}{L}(N_3^2 + L^2).\end{aligned} \tag{256}$$

Proceeding as before, we now find

$$\frac{H}{K} = \frac{N_1N_2 + L^2}{PN_3}, \quad \frac{K}{H} = \frac{N_1N_2 - L^2}{PN_3}, \tag{257}$$

and

$$H^2 - K^2 = 1, \tag{258}$$

$$2HK = \frac{PN_3}{L^2}, \tag{259}$$

$$H^2 + K^2 = \frac{N_1N_2}{L^2}, \tag{260}$$

$$H^2 = \frac{H_1N_2 + L^2}{2L^2},$$

$$K^2 = \frac{N_1N_2 - L^2}{2L^2}, \tag{261}$$

$$\frac{H_1}{K} = \frac{(L-P)^2 + N_1N_2}{PM},$$

$$\frac{K_1}{H} = \frac{(L-P)^2 - N_1N_2}{PM}, \tag{262}$$

and now all the conditions are found to be satisfied.

28. Both results for the prolate and oblate body with parameter

$$v = \omega_1 + \tfrac{1}{2}\omega_3, \text{ or } \tfrac{1}{2}\omega_3, \tag{263}$$

can be incorporated in the form

$$\chi = \tfrac{1}{2}\cos^{-1}\frac{Hz + H_1}{1 - z^2}\sqrt{(z - z_0 \,.\, z - z_3)} \text{ or } \sqrt{(z - z_1 \,.\, z - z_2)}$$

$$= \tfrac{1}{2}\sin^{-1}\frac{Kz + K_1}{1 - z^2}\sqrt{(z - z_1 \,.\, z - z_2)} \text{ or } \sqrt{(z_3 - z \,.\, z - z_0)}, \tag{264}$$

with

$$H^2 + aK^2 = 1, \tag{265}$$

$$H^2 - aK^2 = -a\frac{N_1N_2}{L^2}, \tag{266}$$

$$2HK = \frac{PN_3}{L^2}. \tag{267}$$

If we try to cancel the secular term pt by putting

$$L = \tfrac{1}{2}P = \tfrac{1}{2}, \text{ or } \tfrac{1}{2}(1 + 2c), \tag{268}$$

we find

$$N_1^2 = -\tfrac{3}{4} - c, \text{ or } N_3^2 = -\tfrac{1}{4}, \tag{269}$$

thus leading to imaginary results.

As verifications we may take the numerical case for a parameter $v = \omega_1 + \frac{1}{2}\omega_3$, worked out on p. 534 of "Les fonctions elliptiques et leur applications," Greenhill-Griess, 1895.

For a parameter $v = \frac{1}{2}\omega_3$, take, as a special case,

$$B = 0, \quad N_3 = 0, \quad N_1 = 1 + c, \quad N_2 = c,$$
$$L = -\sqrt{(c + c^2)}, \quad M = \sqrt{(1 + 2c)}\{\sqrt{(1 + c)} + \sqrt{c}\},$$
$$\frac{p}{n} = \frac{L - \frac{1}{2}P}{M} = -\frac{\sqrt{(1 + c)} + \sqrt{c}}{2\sqrt{(1 + 2c)}};$$

and now

$$\psi - pt = \tfrac{1}{2}\cos^{-1}\frac{z}{1 - z^2}\sqrt{\left[z^2 - \frac{\{\sqrt{(1 + c)} - \sqrt{c}\}^2}{1 + 2c}\right]}$$
$$= \tfrac{1}{2}\sin^{-1}\frac{\dfrac{\sqrt{(1 + c)} + \sqrt{c}}{\sqrt{(1 + 2c)}}}{1 - z^2}\sqrt{[(1+2c)\{\sqrt{(1+c)} - \sqrt{c}\}^2 - z^2]}. \quad (269a)$$

29. Proceeding to the next case where the parameter

$$v = \omega_1 + \tfrac{1}{3}\omega_3, \text{ or } \tfrac{1}{3}\omega_3, \quad (270)$$

both results for the prolate and the oblate body can be incorporated in the form

$$\chi = \tfrac{1}{3}\cos^{-1}\frac{Hz^2 + H_1 z + H_2}{(1 - z^2)^{\frac{3}{2}}}\sqrt{(z_3 - z \,.\, z - z_1)} \text{ or } \sqrt{(z - z_2 \,.\, z - z_1)}$$
$$= \tfrac{1}{3}\sin^{-1}\frac{Kz^2 + K_1 z + K_2}{(1 - z^2)^{\frac{3}{2}}}\sqrt{(z_0 - z \,.\, z - z_2)} \text{ or } \sqrt{z_3 - z \,.\, z - z_0}). \quad (271)$$

With a parameter

$$v = \omega_1 + \tfrac{2}{3}\omega_3 \quad (272)$$

we should have to interchange the suffixes 3 and 0, in the first form.

In the associated pseudo-elliptic integrals (Proc. London Math. Society, XXV, p. 218), corresponding to the parameters in (270),

$$s_1 = (1 - c)^2, \quad s_2 = c^2, \quad s_3 = (c - c^2)^2; \quad (273)$$
$$P = \tfrac{2}{3}(2 - c)(1 - 2c), \text{ or } \tfrac{2}{3}(1 + c)(2 - c), \quad (274)$$
$$\sigma = 2c(1 - c)^2, \text{ or } -2c + 2c^2, \quad (275)$$
$$\sqrt{-\Sigma} = 2c(1 - c)^2(2 - c)(1 - 2c), \text{ or } 2c(1 - c)(1 + c)(2 - c). \quad (276)$$

Differentiating as before, and equating coefficients, will serve to determine the unknown coefficients H and K; the work, which is very long and laborious, has been carried out for me by Mr. T. I. Dewar, and he has found that

$$K^2 - aH^2 = \tfrac{3}{2}\frac{PN_1N_3}{L^3} \text{ or } \tfrac{3}{2}\frac{PN_1N_2}{L^2}, \tag{277}$$

while squaring and adding in (271) gives

$$K^2 + aH^2 = 1, \tag{278}$$

whence K and H being determined, the other coefficients readily follow.

Thus, for the prolate body, $v = \omega_1 + \frac{1}{3}\omega_3$,

$$H^2 = \frac{L^3 - (2-c)(1-2c)\,N_1N_3}{2L^3},$$

$$K^2 = \frac{L^3 + (2-c)(1-2c)\,N_1N_3}{2L^3}; \tag{279}$$

so that

$$2HK = \frac{L^2 - (1-c)^2(2-c)(1-2c)}{L^3}N_2; \tag{280}$$

while for the oblate body, $v = \frac{1}{3}\omega_3$,

$$H^2 = \frac{(1+c)(2-c)\,N_1N_2 - L^3}{2L^3},$$

$$K^2 = \frac{(1+c)(2-c)\,N_1N_2 - L^3}{2L^3}; \tag{281}$$

so that

$$2HK = \frac{(1+c)(2-c) - L^2}{L^3}N_3. \tag{282}$$

If we try to cancel the secular term by putting

$$L = \tfrac{1}{2}P = \tfrac{1}{3}(2-c)(1-2c), \text{ or } \tfrac{1}{3}(1+c)(2-c), \tag{283}$$

we find

$$9N_1^2 = (1-2c)(1-c)(-5+2c+2c^2), \text{ or } 9N_3^2 = -2(1+c)(2-c)(1-2c)^2, \tag{284}$$

and these are negative for the region $0 < c < 1$, so that algebraical cases cannot be constructed.

As a numerical verification with a parameter $v = \omega_1 + \frac{1}{3}\omega_3$, we may take the case worked out in the "Applications of Elliptic Functions," p. 348, in which

$$\psi - nt = \tfrac{1}{3}\cos^{-1}\frac{\sqrt{7}z^2 - 4z + \sqrt{7}}{2\sqrt{2}(1-z^2)^{\frac{3}{2}}}\sqrt{(-z^2 - 2\sqrt{7}z + 5)}$$

$$= \tfrac{1}{3}\sin^{-1}\frac{(-z^2 + 2\sqrt{7}z - 3)^{\frac{3}{2}}}{2\sqrt{2}(1-z^2)^{\frac{3}{2}}}. \tag{284a}$$

With $B = 0$ and a parameter $v = \omega_1 + \frac{1}{3}\omega_3$,

$$N_1 = 0, \quad N_2 = \sqrt{(1 - 2c)}, \quad N_3 = (1 - c)\sqrt{(1 - c^2)},$$

$$L = \sqrt{(1 - 2c \,.\, 1 - c^2)}, \quad M = c\sqrt{(1 - c^2)} - c\sqrt{(1 - 2c)},$$

$$P = \tfrac{2}{3}(1 + c)(1 - 2c),$$

$$\psi - pt = \tfrac{1}{3}\cos^{-1}\frac{Hz^2 + H_1 z + H_2}{(1 - z^2)^{\frac{3}{2}}}\sqrt{\left\{\left(\frac{N_2 - N_3}{M}\right)^2 - z^2\right\}}$$

$$= \tfrac{1}{3}\sin^{-1}\frac{Kz^2 + K_1 z + K_2}{(1 - z^2)^{\frac{3}{2}}}\sqrt{\left\{\left(\frac{N_2 + N_3}{M}\right)^2 - z^2\right\}}, \qquad (284b)$$

in which $H = 0, \quad K = 1, \quad K_1 = 0, \quad H_2 = 0,$

$$H_1 = 3\,\frac{p}{n} = \frac{(1 + c)\sqrt{(1 - 2c)} + (1 - 2c)\sqrt{(1 - c^2)}}{c^2},$$

$$K_2 = \frac{-M}{N_2 - N_3} = -\tfrac{1}{2}\left\{\frac{\sqrt{(1 - c)} - \sqrt{(1 + c \,.\, 1 - 2c)}}{c}\right\}^2.$$

With $B = 0$, and a parameter $v = \frac{1}{3}\omega_3$,

$$N_3 = 0, \quad L = -\sqrt{(1 - c^2 \,.\, 2c - c^2)},$$

$$N_1^2 = (1 - c)^3(1 + c), \quad N_2^2 = c^2(2c - c^2),$$

$$M = \sqrt{(1 - c^2)} + \sqrt{(2c - c^2)},$$

$$\psi - pt = \tfrac{1}{3}\cos^{-1}\frac{Hz^2 + H_1 z + H_2}{(1 - z^2)^{\frac{3}{2}}}\sqrt{\left\{z^2 - \left(\frac{N_1 - N_2}{M}\right)^2\right\}}$$

$$= \tfrac{1}{3}\sin^{-1}\frac{Kz^2 + K_1 z + K_2}{(1 - z^2)^{\frac{3}{2}}}\sqrt{\left\{\left(\frac{N_1 + N_2}{M}\right)^2 - z^2\right\}}, \qquad (284c)$$

in which

$$H = 0, \quad H_1 = -3\,\frac{p}{n} = (2 - c)\sqrt{(1 - c^2)} + (1 + c)\sqrt{(2c - c^2)}, \quad H_2 = 0;$$

$$K = 1, \quad K_1 = 0, \quad K_2 = \frac{M}{N_1 + N_2} = \tfrac{1}{2}\{\sqrt{(1 - c^2)} + \sqrt{(2c - c^2)}\}^2.$$

30. These algebraical calculations are too long and complicated to be inserted here, and the complexity compelled us, as in the corresponding work for the Spherical Catenary, in the Proc. London Math. Society, XXVII, to turn elsewhere for some clue to the values of the leading coefficients H and K, upon which the remainder depend by simple linear relations.

As in the Spherical Catenary, we turn to the degenerate form assumed when we take

$$z = \infty, \quad u = v_3.$$

The preceding cases enable us to infer that the general form of the solution for a parameter

$$v = \omega_1 + \frac{2\omega_3}{n}, \text{ or } \omega_1 + \frac{\omega_3}{n} \text{ or } \frac{\omega_3}{n}, \tag{285}$$

where n is odd, can be written

$$\begin{aligned}\chi &= \frac{1}{n}\cos^{-1}\frac{Hz^{n-1} + \dots.}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z_0 - z \, . \, z - z_1)}, \text{ or } \sqrt{(z_3 - z \, . \, z - z_1)}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{or } \sqrt{(z - z_1 \, . \, z - z_2)} \\ &= \frac{1}{n}\sin^{-1}\frac{Kz^{n-1} + \dots.}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z_3 - z \, . \, z - z_2)}, \text{ or } \sqrt{(z_0 - z \, . \, z - z_2)}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{or } \sqrt{(z_3 - z \, . \, z - z_0)},\end{aligned} \tag{286}$$

but for parameters

$$v = \omega_1 + \frac{\omega_3}{n}, \text{ or } \frac{\omega_3}{n}, \tag{287}$$

and n even, we must take

$$\begin{aligned}\chi &= \frac{1}{n}\cos^{-1}\frac{Hz^{n-1} + \dots.}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z_0 - z \, . \, z_3 - z)}, \text{ or } \sqrt{(z - z_1 \, . \, z - z_2)} \\ &= \frac{1}{n}\sin^{-1}\frac{Kz^{n-1} + \dots.}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z - z_2 \, . \, z - z_1)}, \text{ or } \sqrt{(z_3 - z \, . \, z - z_0)}.\end{aligned} \tag{288}$$

Squaring and adding in (286) and (288) shows that

$$H^2 + K^2 = 1, \text{ or } H^2 - K^2 = (-1)^n, \tag{289}$$

according as the body is prolate or oblate; and putting $z = \infty$,

$$\begin{aligned}\chi &= \frac{1}{n}\tan^{-1}\frac{K\sqrt{a}}{H} = \frac{1}{2n}\sin^{-1} 2HK\sqrt{a} \\ &= \frac{1}{2n}\cos^{-1}(H^2 - K^2), \text{ or } \frac{1}{2n}\cos^{-1}(H^2 + K^2)(-1)^n.\end{aligned} \tag{290}$$

But, by means of the formula

$$\frac{\wp(u - v_1) - \wp(u - v_2)}{\wp(u + v_1) - \wp(u + v_2)} = \frac{\sigma(2u - v_1 - v_2)\,\sigma^2(u + v_1)\,\sigma^2(u + v_2)}{\sigma(2u + v_1 + v_2)\,\sigma^2(u - v_1)\,\sigma^2(u - v_2)}, \tag{291}$$

we may write equation (133)

$$\begin{aligned}\psi i &= \tfrac{1}{2}\,Qnt + \tfrac{1}{4}\log\frac{\sigma(2u + v)}{\sigma(2u - v)} + \tfrac{1}{4}\log\frac{\wp(u - v_1) - \wp(u - v_2)}{\wp(u + v_1) - \wp(u + v_2)} \\ &= ipt + \tfrac{1}{2}\,i\,I(2u, v) \qquad\quad + \tfrac{1}{4}\log\frac{\wp(u - v_1) - \wp(u - v_2)}{\wp(u + v_1) - \wp(u + v_2)},\end{aligned} \tag{292}$$

thus adding the amplitudes and parameters of the two Elliptic Integrals of the Third Kind in ψ_1 and ψ_2.

Putting $z = \infty$ and $u = v_3$ in this last relation, then from (96), (97),

$$\frac{\wp(v_3 - v_1) - \wp(v_3 - v_2)}{\wp(v_3 + v_1) - \wp(v_3 + v_2)} = -1, \tag{293}$$

while $I(2v_3, v)$ is the value of the pseudo-elliptic integral when $u = 2v_3$ or

$$\frac{s - \sigma}{M^2} = \wp 2v_3 - \wp(v_1 + v_2) = \frac{L^2}{M^2}, \tag{294}$$

$$s - \sigma = L^2, \tag{295}$$

$$s - s_a = aN_a^2. \tag{296}$$

The form of $I(v)$ being given by

$$\begin{aligned} I(v) &= \frac{1}{n}\sin^{-1}\frac{F\sqrt{(s - s_a)}}{(s - \sigma)^{\frac{1}{2}n}} \\ &= \frac{1}{n}\cos^{-1}\frac{G\sqrt{(s - s_\beta \,.\, s - s_\gamma)}}{(s - \sigma)^{\frac{1}{2}n}}, \end{aligned} \tag{297}$$

therefore, when $z = \infty$ and $\psi - pt$ is replaced by χ,

$$\chi = \tfrac{1}{2} I(2v_3, v) + \tfrac{1}{4} i \log(-1); \tag{298}$$

or, disregarding for a moment the ambiguities of sign,

$$\frac{1}{2n}\sin^{-1}\frac{FN_a\sqrt{a}}{L^n} = \frac{1}{2n}\sin^{-1} 2HK\sqrt{a}, \tag{299}$$

$$\frac{1}{2n}\cos^{-1}\frac{GN_\beta N_\gamma}{L^n} = \frac{1}{2n}\cos^{-1}(H^2 - aK^2), \tag{300}$$

or

$$2HK = \frac{FN_a}{L^n} \tag{301}$$

$$H^2 - aK^2 = \pm\frac{GN_\beta N_\gamma}{L^n}, \tag{302}$$

and these equations, together with (289), determine the leading coefficients H and K, upon which the other coefficients depend by simple linear relations.

Since $z = \infty$ in (167) makes $\varpi = \psi$, we see that the leading coefficients I and J in (184) are determined at the same time, by taking $I = K$ and $J = H$ or $I = H$ and $J = K$, according to circumstances.

31. To make quite sure of the plus and minus signs, which are apt to be baffling, it is well to make a recapitulation of all the various cases which may occur, according as the body is prolate or oblate, and n is even or odd.

When the body is prolate, and the parameter of the form

$$v = \omega_1 + \frac{\omega_3}{n}, \tag{303}$$

then, when n is even, we must take

$$\chi = \frac{1}{n} \cos^{-1} \frac{Hz^{n-1} + H_1 z^{n-2} + \dots + H_{n-1}}{(1-z^2)^{\frac{1}{2}n}} \sqrt{(z_0 - z \,.\, z_3 - z)}$$
$$= \frac{1}{n} \sin^{-1} \frac{Kz^{n-1} + K_1 z^{n-2} + \dots + K_{n-1}}{(1-z^2)^{\frac{1}{2}n}} \sqrt{(z - z_2 \,.\, z - z_1)}, \tag{304}$$

and when n is odd,

$$\chi = \frac{1}{n} \cos^{-1} \frac{Hz^{n-1} \qquad + \dots + H_{n-1}}{(1-z^2)^{\frac{1}{2}n}} \sqrt{(z_3 - z \,.\, z - z_1)}$$
$$= \frac{1}{n} \sin^{-1} \frac{Kz^{n-1} \qquad + \dots + K_{n-1}}{(1-z^2)^{\frac{1}{2}n}} \sqrt{(z_0 - z \,.\, z - z_2)}. \tag{305}$$

But with a parameter

$$v = \omega_1 + \frac{2\omega_3}{n}, \tag{306}$$

where n is an odd number,

$$\chi = \frac{1}{n} \cos^{-1} \frac{Hz^{n-1} + \dots}{(1-z^2)^{\frac{1}{2}n}} \qquad \sqrt{(z_0 - z \,.\, z - z_1)}$$
$$= \frac{1}{n} \sin^{-1} \frac{Kz^{n-1} + \dots}{(1-z^2)^{\frac{1}{2}n}} \qquad \sqrt{(z_3 - z \,.\, z - z_2)}, \tag{307}$$

obtainable from the preceding case (306) by an interchange of z_0 and z_3, or z_1 and z_2.

In all these cases of the prolate body

$$z_3 > z > z_2, \tag{308}$$

and
$$H^2 + K^2 = 1. \tag{309}$$

Putting $z = \infty$,

$$\chi = \frac{1}{n} \tan^{-1} \frac{K}{H} = \frac{1}{2n} \sin^{-1} 2HK = \frac{1}{2n} \cos^{-1} (H^2 - K^2), \tag{310}$$

and at the same time

$$\chi = \tfrac{1}{2} I(2v_3, v) = \frac{1}{2n} \sin^{-1} \frac{Fn_\alpha}{L^n} = \frac{1}{2n} \cos^{-1} \frac{GN_\beta N_\gamma}{L^n}; \tag{311}$$

so that

$$2HK = \frac{FN_\alpha}{L^n}, \tag{312}$$

$$H^2 - K^2 = \frac{GN_\beta N_\gamma}{L^n}. \tag{313}$$

32. When the body is oblate and the parameter is of the form

$$v = \frac{\omega_3}{n}, \tag{314}$$

we must take

$$\chi = \frac{1}{n}\cos^{-1}\frac{Hz^{n-1} + \dots + H_{n-1}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z_3 - z \,.\, z - z_0)}$$

$$= \frac{1}{n}\sin^{-1}\frac{Kz^{n-1} + \dots + K_{n-1}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{(z - z_2 \,.\, z - z_1)} \tag{315}$$

and

$$H^2 - K^2 = (-1)^n. \tag{316}$$

Putting $z = \infty$ and $u = v_3$,

$$\chi = \frac{1}{n}\tan^{-1}\frac{Ki}{H} = \frac{1}{2n}\sin^{-1} 2HKi(-1)^n = \frac{1}{2n}\cos^{-1}(H^2 + K^2)(-1)^n, \tag{317}$$

so that, allowing for the effect of the $\frac{1}{4}i\log(-1)$ in (298), we may put

$$2HK = \frac{FN_\alpha}{L^n}, \tag{318}$$

$$H^2 + K^2 = \frac{GN_\alpha}{L^n}. \tag{319}$$

33. Finally in the case of an oblate body, when the parameter

$$v = \frac{2\omega_3}{n}, \tag{320}$$

and n is an odd number, we must take

$$\chi = \frac{1}{n}\cos^{-1}\frac{Hz^n + H_1 z^{n-1} + \dots + H_n}{(1-z^2)^{\frac{1}{2}n}}$$

$$= \frac{1}{n}\sin^{-1}\frac{Kz^{n-2} + K_1 z^{n-3} + \dots + K_{n-2}}{(1-z^2)^{\frac{1}{2}n}}\sqrt{Z}, \tag{321}$$

with

$$K^2 - H^2 = 1; \tag{322}$$

and proceeding to $z = \infty$,

$$\chi = \frac{1}{n}\tan^{-1}\frac{Ki}{H} = \frac{1}{2n}\cos^{-1}(H^2 + K^2). \tag{323}$$

At the same time the associated pseudo-elliptic integral

$$I(2v_3,\ v) = \frac{1}{n} \cos^{-1} \frac{F}{L^n} = \frac{1}{n} \sin^{-1} \frac{GN_\alpha N_\beta N_\gamma i}{L^n}, \tag{324}$$

so that

$$H^2 + K^2 = \frac{F}{L^n}, \tag{325}$$

$$2HK = \frac{GN_\alpha N_\beta N_\gamma}{L^n}. \tag{326}$$

In such cases it is generally simpler to take a parameter

$$v = \frac{4\omega_3}{n} \tag{327}$$

so as to make $\sigma = 0$ in (294), and now, when $z = \infty$,

$$s = L^2, \quad N_\alpha^2 = s_\alpha - L^2. \tag{328}$$

The resolution of the cubic S in (50) is not required now, because symmetric functions only of the roots s_1, s_2, s_3 occur; thus from (63),

$$\begin{aligned} M^2 &= N_1^2 + N_2^2 + N_3^3 + 2L^2 - \frac{\sqrt{-\Sigma}}{L} \\ &= s_1 + s_2 + s_3 - L^2 - \frac{xy}{L} \\ &= \tfrac{1}{4}(y+1)^2 - 2x - L^2 - \frac{xy}{L}, \end{aligned} \tag{329}$$

$$\begin{aligned} LM^2 &= -xy + \tfrac{1}{4}\{(y+1)^2 - 8x\} L - L^3 \\ &= M^3 i\wp' v + 3LM^2 \wp v - L^3, \end{aligned} \tag{330}$$

as in Lamé's equation of the second order.

34. With $s = L^2$,

$$-S = \{xy + (y+1)L^2\}^2 - 4L^2(x + L^2)^2 = S_1 S_2, \tag{331}$$

where

$$\begin{aligned} S_1 &= xy + 2xL + (y+1)L^2 + 2L^3, \\ S_2 &= xy - 2xL + (y+1)L^2 - 2L^3. \end{aligned} \tag{332}$$

At the same time we shall find that we can put

$$\begin{aligned} \tfrac{1}{2} I(2v_3,\ v) &= \frac{1}{n} \sin^{-1} \frac{A + BL + CL^2 \ldots .}{2L^{\frac{1}{2}n}} \sqrt{S_1} \\ &= \frac{1}{n} \cos^{-1} \frac{A - BL + CL^2 \ldots .}{2L^{\frac{1}{2}n}} \frac{\sqrt{S_2}}{i}; \end{aligned} \tag{333}$$

while with $z = \infty$,

$$\chi = \frac{1}{n} \sin^{-1} K = \frac{1}{n} \cos^{-1} \frac{H}{i}, \tag{334}$$

so that

$$K \text{ or } H = \frac{A + BL + CL^2 + \dots}{2L^{\frac{1}{2}n}, \text{ or } 2(-L)^{\frac{1}{2}n}} \surd S_1,$$

$$H \text{ or } K = \frac{A - BL + CL^2 - \dots}{2L^{\frac{1}{2}n} \text{ or } 2(-L)^{\frac{1}{2}n}} \surd S_2, \tag{335}$$

thus determining the leading coefficients, when the parameter $v = 4\omega_3/n$.

35. For instance, with $n = 3$, and taking the corresponding pseudo-elliptic integral, with $P = \frac{1}{3}$,

$$I(\tfrac{4}{3}\omega_3) = \tfrac{1}{2} \int \frac{\frac{1}{3}s + c}{s\surd\{4s^3 - (s+c)^2\}}$$

$$= \tfrac{1}{3} \cos^{-1} \frac{s - c}{2s^{\frac{3}{2}}} = \tfrac{1}{3} \sin^{-1} \frac{\surd S}{2s^{\frac{3}{2}}}, \tag{336}$$

we have

$$K = \frac{\surd(c + L^2 + 2L^3)}{2L^{\frac{3}{2}}},$$

$$H = \frac{\surd(c + L^2 - 2L^3)}{2L^{\frac{3}{2}}}. \tag{337}$$

Then

$$LM^2 = c + \tfrac{1}{4}L - L^3, \tag{338}$$

$$\frac{B}{M} = -\frac{N_1 N_2 N_3}{LM^2} = \frac{\frac{1}{2}\surd\{(c + L^2)^2 - 4L^6\}}{c + \frac{1}{4}L - L^3}$$

$$= 2\frac{\surd\{(c + L^2)^2 - 4L^6\}}{4c + L - 4L^3}, \tag{339}$$

and the result is of the form

$$\psi - pt = \tfrac{1}{3} \cos^{-1} \frac{Hz^3 + H_1 z^2 + H_2 z + H_3}{(1 - z^2)^{\frac{3}{2}}} = \tfrac{1}{3} \sin^{-1} \frac{Kz + K_1}{(1 - z^2)^{\frac{3}{2}}} \surd Z. \tag{340}$$

By squaring and adding, and by differentiation, we find

$$H_1 = \frac{1 - 6L}{2M} K,$$

$$K_1 = -\frac{H}{K} H_1 = -\frac{1 - 6L}{M} H,$$

$$H_2 = -\frac{1 + 6L}{2M} K_1 = \frac{1 - 36L^2}{2M^2} H,$$

$$H_3 = \frac{2L - P}{M} K - \frac{2L + P}{2M} K + 2\frac{B}{M} K_1,$$

$$= \frac{2L - 3P}{2M} K + 2\frac{B}{M} K_1$$

$$= -\frac{1 - 2L}{2M} K - 4\frac{\surd\{(c + L^2)^2 - 4L^6\}}{4c + L - 4L^3} \frac{1 - 6L}{M} H$$

$$= -\left\{\frac{1 - 2L}{M} + 4\frac{c + L^2 - 2L^3}{4c + L - 4L^3} \frac{1 - 6L}{M}\right\} K$$

$$= -\left\{1 - 2L + 4(1 - 6L)\frac{c + L^2 - 2L^3}{4c + L - 4L^3}\right\} \frac{K}{M}. \tag{341}$$

If we cancel the secular term by putting

$$L = \tfrac{1}{2} P = \tfrac{1}{6}, \tag{342}$$

then

$$H_1 = 0, \quad K_1 = 0, \quad H_2 = 0,$$

and

$$H_3 = -\tfrac{1}{3}\frac{K}{M},$$

$$H = \sqrt{(54c+1)},$$

$$K = \sqrt{(54c+2)},$$

$$9M^2 = 54c + 2 = K^2,$$

$$3M = -K, \quad H_3 = -1,$$

$$\frac{B}{M} = \tfrac{1}{2}\frac{H}{K}, \quad \frac{L}{M} = \frac{1}{2K}, \tag{343}$$

and now

$$\psi = \tfrac{1}{3}\cos^{-1}\frac{\sqrt{(54c+1)}\,z^3 - 1}{(1-z^2)^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\sin^{-1}\frac{z}{(1-z^2)^{\frac{3}{2}}}\sqrt{\{-(54c+2)z^4 + 3z^2 + 2\sqrt{(54c+1)}\,z - 3\}}. \tag{344}$$

This can also be written, with H for $\sqrt{(54c+1)}$,

$$(1-z^2)^{\frac{3}{2}} e^{3\psi i} = Hz^3 - 1 + iz\sqrt{\{-(H^2+1)z^4 + 3z^2 + 2Hz - 3\}}, \tag{345}$$

which by differentiation will be found to verify the relation

$$\frac{d\psi}{dz} = 2\frac{L - Bz}{M(1-z^2)\sqrt{Z_1}}, \tag{346}$$

and thus (345) represents an algebraical case of the motion of the axis OC of an oblate body, relatively to a line Oz fixed in direction.

At the same time the curve of (α, β) or (ρ, ϖ) will be given by a relation of the form

$$\varpi = \tfrac{1}{3}\cos^{-1}\frac{Iz^3 + I_1z^2 + I_2z + I_3}{(z^2 - 1 + E)^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\sin^{-1}\frac{Jz + J_1}{(z^2 - 1 + E)^{\frac{3}{2}}}\sqrt{Z}, \tag{347}$$

where

$$I = K = \sqrt{(H^2+1)} \text{ and } J = H, \tag{348}$$

or

$$\left(\frac{F}{An}\rho e^{\varpi i}\right)^3 = \left\{\frac{F}{An}(\alpha + \beta i)\})\right\}^3$$

$$= Iz^3 + I_1z^2 + I_2z + I_3 + i(Jz + J_1)\sqrt{Z}, \tag{349}$$

and we shall find that with

$$E = 2\frac{H^2-1}{H^2+1}, \quad I_1 = \frac{3H}{\sqrt{(H^2+1)}}, \quad I_2 = 0, \quad I_3 = -\frac{H^3+9H}{(H^2+1)^{\frac{3}{2}}}, \quad J_1 = 3, \tag{350}$$

the differential relation of (169) can be satisfied.

The discriminant of

$$Z_1 = -(H^2+1)z^4 + 3z^2 + 2Hz - 3 \tag{351}$$

is

$$-\tfrac{27}{16}(H^2-1)^3(H^2+1). \tag{352}$$

If $H^2 < 1$, all four roots of this quartic are imaginary, and Z is always negative, so that no real solution exists.

But if $H^2 > 1$, the discriminant is negative, and Z has two real roots, so that real cases of motion can be constructed.

36. Making use of the results of the pseudo-elliptic integrals of higher orders, we find for $n = 5$, putting Halphen's $y = x$,

$$K \text{ or } H = \frac{(x-L)\sqrt{\{x^2 + 2xL + (1+x)L^2 + 2L^3\}}}{2L^{\frac{5}{2}}, \text{ or } 2(-L)^{\frac{5}{2}}},$$

$$H \text{ or } K = \frac{(x+L)\sqrt{\{x^2 - 2xL + (1+x)L^2 - 2L^3\}}}{2L^{\frac{5}{2}} \text{ or } 2(-L)^{\frac{5}{2}}}, \tag{353}$$

according as L is positive or negative, and to make the solution algebraical by cancelling the secular term, put

$$L = \tfrac{1}{2}P(v) = \frac{1-3x}{10}. \tag{354}$$

This makes

$$K \text{ or } H = \frac{(-1+13x)\sqrt{(3+33x+121x^2+9x^3)}}{(1-3x)^{\frac{5}{2}} \text{ or } (3x-1)^{\frac{5}{2}}},$$

$$H \text{ or } K = \frac{(1+7x)\sqrt{\{2(1-18x)(1-11x-x^2)}}{(1-3x)^{\frac{5}{2}} \text{ or } (3x-1)^{\frac{5}{2}}}; \tag{355}$$

$$N_1N_2N_3 = \sqrt{\{(s_1-L^2)(s_2-L^2)(s_3-L^2)\}}$$

$$= \frac{\sqrt{\{2(1-18x)(1-11x-x^2)(3+33x+121x^2+9x^3)\}}}{1000}, \tag{356}$$

$$LM^2 = \frac{24(1+2x)(1-11x-x^2)}{1000}, \tag{357}$$

and putting

$$\psi = \tfrac{1}{5}\cos^{-1}\frac{Hz^5 + H_1z^4 + \dots + H_5}{(1-z^2)^{\frac{5}{2}}}$$

$$= \tfrac{1}{5}\sin^{-1}\frac{Kz^3 + K_1z^2 + K_2 + K_3}{(1-z^2)^{\frac{5}{2}}}\sqrt{Z}, \tag{358}$$

where
$$Z = -(z^2-1)(z^2-1+E) - 4\left(\frac{Lz-B}{M}\right)^2, \tag{359}$$
the differential relation (46) shows at once that
$$H_1 = 0 \text{ and } K_1 = 0. \tag{360}$$

As the result of an algebraical verification, it will be found that with

$$\begin{aligned}
L &= -\frac{3x-1}{10}, \\
LM^2 &= -\frac{24(2x+1)(x^2+11x-1)}{1000}, \\
M^2 &= \frac{24(2x+1)(x^2+11x-1)}{100(3x-1)}, \\
\frac{L^2}{M^2} &= \frac{(3x-1)^3}{24(2x+1)(x^2+11x-1)}, \\
\frac{BL}{M^2} &= -\frac{N_1N_2N_3}{M^3} \\
&= -\frac{1}{24\sqrt{6}}\,\frac{(3x-1)^{\frac{3}{2}}\sqrt{\{2(18x-1)(9x^3+121x^2+33x+3)\}}}{(2x+1)^{\frac{3}{2}}\qquad(x^2+11x-1)}, \\
\frac{B}{M} &= \frac{1}{12}\,\frac{\sqrt{\{2(18x-1)(9x^3+121x^2+33x+3)\}}}{(2x+1)\sqrt{(x^2+11x-1)}}, \\
\frac{L}{M} &= -\frac{1}{2\sqrt{6}}\,\frac{(3x-1)^{\frac{3}{2}}}{\sqrt{\{(2x+1)(x^2+11x-1)\}}}, \\
E &= -\frac{2x^2}{LM^2} = \frac{250x^2}{3(2x+1)(x^2+11x-1)}, \\
k^2 &= 1-E = \frac{6x^3-181x^2+27x-3}{3(2x+1)(x^2+11x-1)};
\end{aligned} \tag{361}$$

and then

$$\begin{aligned}
Z = -z^4 &- \frac{(3x-1)(x^2+66x-11)}{6(2x+1)(x^2+11x-1)}z^2 \\
&- \frac{1}{3\sqrt{6}}\,\frac{(3x-1)^{\frac{3}{2}}\sqrt{\{2(18x-1)(9x^3+121x^2+33x+3)\}}}{(2x+1)^{\frac{3}{2}}\qquad(x^2+11x-1)}z \\
&- \frac{1}{18}\,\frac{(3x-1)^2(26x^2+21x-21)}{(2x+1)^2(x^2+11x-1)}.
\end{aligned} \tag{362}$$

We must now take

$$\begin{aligned}
H &= \frac{(13x-1)\sqrt{(9x^3+121x^2+33x+3)}}{(3x-1)^{\frac{5}{2}}}, \\
K &= \frac{(7x+1)\sqrt{\{2(18x-1)(x^2+11x-1)\}}}{(3x-1)^{\frac{5}{2}}};
\end{aligned} \tag{363}$$

and thence we find

$$H_2 = \frac{5}{3}\,\frac{(8x-1)\sqrt{(9x^3+121x^2+33x+3)}}{(2x+1)(3x-1)^{\frac{3}{2}}},$$

$$K_2 = \frac{1}{3}\,\frac{(12x+1)\sqrt{\{2(18x-1)(x^2+11x-1)\}}}{(2x+1)(3x-1)^{\frac{3}{2}}},$$

$$H_3 = \frac{5}{3\sqrt{6}}\,\frac{(7x+1)\sqrt{\{2(18x-1)}}{(2x+1)^{\frac{1}{2}}(3x-1)},$$

$$K_3 = \frac{4}{3\sqrt{6}}\,\frac{\sqrt{(9x^3+121x^2+33x+3)}\sqrt{(x^2+11x-1)}}{(2x+1)^{\frac{3}{2}}(3x-1)},$$

$$H_4 = \frac{10}{9}\,\frac{\sqrt{(3x-1)}\sqrt{(9x^3+121x^2+33x+3)}}{(2x+1)^2},$$

$$H_5 = \frac{1}{9\sqrt{6}}\,\frac{(22x^2+42x+3)\sqrt{\{2(18x-1)\}}}{(2x+1)^{\frac{5}{2}}}, \qquad (364)$$

A numerical verification is obtained by taking $x = 2$; this makes

$$L = -\tfrac{1}{2},\quad M = \sqrt{6},\quad \frac{L}{M} = -\frac{\sqrt{6}}{12},\quad \frac{B}{M} = \frac{\sqrt{70}}{12},$$
$$E = \tfrac{8}{3},\quad k^2 = 1 - E = -\tfrac{5}{3}; \qquad (365)$$

and then

$$Z = -z^4 - \frac{5}{6}z^2 - \frac{\sqrt{105}}{9}z - \frac{5}{18}; \qquad (366)$$

and

$$H = 5\sqrt{5},\quad K = 3\sqrt{14};$$
$$H_2 = 5\sqrt{5},\quad K_2 = \tfrac{5}{3}\sqrt{14};$$
$$H_3 = \tfrac{5}{3}\sqrt{21},\quad K_3 = \tfrac{2}{9}\sqrt{30};$$
$$H_4 = \tfrac{10}{9}\sqrt{5},\quad H_5 = \tfrac{7}{27}\sqrt{21}. \qquad (367)$$

37. With $n = 7$ and a parameter

$$v = \tfrac{4}{7}\omega_3, \qquad (368)$$

we take, in (49),

$$x = -c(1+c)^2,\quad y = -c(1+c); \qquad (369)$$

and now we find

K or H

$$= \frac{\{c(1+c)^3+(1+c)^2L-L^2\}\sqrt{\{c^2(1+c)^3-2c(1+c)^2L+(1+c-c^2)L^2+2L^3\}}}{2L^{\frac{7}{2}} \text{ or } 2(-L)^{\frac{7}{2}}}$$

H or K

$$= \frac{\{c(1+c)^3-(1+c)^2L-L^2\}\sqrt{\{c^2(1+c)^3+2c(1+c)^2L+(1+c-c^2)L^2-2L^3\}}}{2L^{\frac{7}{2}} \text{ or } 2(-L)^{\frac{7}{2}}}; \qquad (370)$$

and the solution is made algebraical by taking

$$L = \tfrac{1}{2} P(v) = \frac{3 + 9c + 5c^2}{14}. \tag{371}$$

With $n = 9$ and a parameter

$$v = \tfrac{4}{9}\,\omega_3, \tag{372}$$

take ("Pseudo-Elliptic Integrals," p. 232)

$$x = p^2(1-p)(1-p+p^2), \quad y = p^2(1-p); \tag{373}$$

and now

K or $H =$

$$\frac{\{p^6(1-p)(1-p+p^2) - p^4(1-p+p^2)L - p^2(1-2p)L^2 + L^3\} \sqrt{\{p^4(1-p)(1-p+p^2) + 2p^2(1-p)(1-p+p^2)L + (1+0+p^2-p^3)^2 L + 2L^3\}}}{2L^{\frac{3}{2}} \text{ or } 2(-L)^{\frac{3}{2}}},$$

H or $K =$

$$\frac{\{p^4(1-p)(1-p+p^2) + p^4(1-p+p^2)L - p^2(1-2p)L^2 - L^3\} \sqrt{\{p^4(1-p)(1-p+p^2) - 2p^2(1-p)(1-p+p^2)L + (1+0-p^2-p^3)L^2 - 2L^3\}}}{2L^{\frac{3}{2}} \text{ or } 2(-L)^{\frac{3}{2}}}, \tag{375}$$

and so on for the higher values of n, but the complexity increases very rapidly.

38. Making use of the theorem in §30 that $I = K$ or H and $J = H$ or K, according as the body is prolate or oblate, in the associated pseudo-elliptic expressions of ϖ in (184), we can now write down the corresponding formulas, and determine the remaining coefficients by means of the differential relation

$$\frac{d(\varpi - pt)}{dz} = \frac{-\dfrac{L + \frac{1}{2}P}{M} z^2 + 2\dfrac{B}{M} z - \dfrac{L - \frac{1}{2}P}{M}(1 + aE)}{(1 + aE - z^2)\sqrt{Z}}. \tag{192}$$

Thus with the parameter $v = \omega_1 + \frac{1}{2}\omega_3$,

$$\varpi - pt = \tfrac{1}{2}\cos^{-1}\frac{Iz + I_1}{1 + E - z^2}\sqrt{(z_0 - z \,.\, z_3 - z)}$$

$$= \tfrac{1}{2}\sin^{-1}\frac{Jz + J_1}{1 + E - z^2}\sqrt{(z - z_2 \,.\, z - z_1)}; \tag{376}$$

with

$$I^2 = K^2 = \frac{L^2 + N_1 N_2}{2L^2},$$

$$J^2 = H^2 = \frac{L^2 - N_1 N_2}{2L^2}; \tag{377}$$

and then

$$\frac{I_1}{J} = \frac{N_1 N_2 + (L+1)^2}{M}$$

$$\frac{J_1}{I} = \frac{N_1 N_2 - (L+1)^2}{M}. \qquad (378)$$

The relation (376) may also be written

$$\left\{\frac{F}{An}\rho e^{i(\varpi - pt)}\right\}^2 = (Iz + I_1)\sqrt{(z_0 - z \,.\, z - z_3)} + i(Jz + J_1)\sqrt{(z - z_2 \,.\, z - z_1)}. \qquad (379)$$

With a parameter $v = \frac{1}{2}\omega_3$,

$$\varpi - pt = \tfrac{1}{2}\cos^{-1}\frac{Iz + I_1}{z^2 - 1 + E}\sqrt{(z - z_1 \,.\, z - z_2)}$$

$$= \tfrac{1}{2}\sin^{-1}\frac{Jz + J_1}{z^2 - 1 + E}\sqrt{(z_3 - z \,.\, z - z_0)}, \qquad (380)$$

or

$$\left\{\frac{F}{An}\rho e^{i(\varpi - pt)}\right\}^2 = (Iz + I_1)\sqrt{(z - z_1 \,.\, z - z_2)} + i(Jz + J_1)\sqrt{(z_3 - z \,.\, z - z_0)}, \qquad (381)$$

with

$$I^2 = H^2 = \frac{N_1 N_2 - L^2}{2L^2},$$

$$J^2 = K^2 = \frac{N_1 N_2 + L^2}{2L^2}; \qquad (382)$$

and then

$$\frac{I_1}{J} = \frac{(L+P)^2 + N_1 N_2}{PM},$$

$$\frac{J_1}{I} = \frac{(L+P)^2 - N_1 N_2}{LM}. \qquad (383)$$

So also with a parameter

$$v = \omega_1 + \tfrac{2}{3}\omega_3, \text{ or } \omega_1 + \tfrac{1}{3}\omega_3, \text{ or } \tfrac{1}{3}\omega_3, \qquad (270)$$

the expressions take the form

$$\left\{\frac{F}{An}\rho e^{i(\varpi - pt)}\right\}^3$$

$$= (Iz^2 + I_1 z + I_2)\sqrt{(z_0 - z \,.\, z - z_1)} + i(Jz^2 + J_1 z + J_2)\sqrt{(z_3 - z \,.\, z - z_2)}, \qquad (384)$$

or $$= (Iz^2 + I_1 z + I_2)\sqrt{(z_3 - z \,.\, z - z_1)} + i(Jz^2 + J_1 z + J_2)\sqrt{(z_0 - z \,.\, z - z_2)}, \qquad (385)$$

or $$= (Iz^2 + I_1 z + I_2)\sqrt{(z - z_1 \,.\, z - z_2)} + i(Jz^2 + J_1 z + J_2)\sqrt{(z_3 - z \,.\, z - z_0)}. \qquad (386)$$

Thus, in completion of the preceding special cases of §§28, 29, we find for a parameter $v = \omega_1 + \frac{1}{2}\omega_3$, and, in addition, with

$$B = 0,\ N_1 = 0,\ L = \sqrt{(1+c)},$$
$$M = \sqrt{c+c^2} - \sqrt{c};\ N_2 = \sqrt{(1+2c)},\ N_3 = 1+c,$$
$$1 + E = \left\{\frac{\sqrt{(1+c)}+1}{\sqrt{(1+c)}-1}\right\}^2 = k^2,$$

$$\frac{N_2^2 - N_3^2}{M^2} = z_0 z_3 = z_1 z_2 = \frac{\sqrt{(1+c)}+1}{\sqrt{(1+c)}-1} = k;$$

$$\varpi - pt = \tfrac{1}{2}\cos^{-1}\frac{z + k^{\frac{3}{2}}}{\sqrt{2(k^2 - z^2)}}\sqrt{\left(z^2 - 2\frac{N_3}{M}z + k\right)}$$
$$= \tfrac{1}{2}\sin^{-1}\frac{z - k^{\frac{3}{2}}}{\sqrt{2(k^2 - z^2)}}\sqrt{\left(z^2 + 2\frac{N_3}{M}z + k\right)}; \tag{386a}$$

with a corresponding

$$\psi - pt = \tfrac{1}{2}\cos^{-1}\frac{z + \dfrac{\sqrt{(1+c)}-1}{\sqrt{c}}}{\sqrt{2(1-z^2)}}\sqrt{\left(z^2 - 2\frac{N_3}{M}z + k\right)}$$
$$= \tfrac{1}{2}\sin^{-1}\frac{z - \dfrac{\sqrt{(1+c)}-1}{\sqrt{c}}}{\sqrt{2(1-z^2)}}\sqrt{\left(z^2 + 2\frac{N_3}{M}z + k\right)}. \tag{386b}$$

For the parameter $v = \frac{1}{2}\omega_3$, with

$$B = 0,\ N_3 = 0,\ k^2 = 1 - E = \{\sqrt{(1+c)} - \sqrt{c}\}^4,\ L = -\sqrt{(c+c^2)},$$

$$\varpi - pt = \tfrac{1}{2}\cos^{-1}\frac{z}{z^2 - \{\sqrt{(1+c)} - \sqrt{c}\}^4}\sqrt{\left[z^2 - \frac{\{\sqrt{(1+c)} - \sqrt{c}\}^2}{1+2c}\right]}$$
$$= \tfrac{1}{2}\sin^{-1}\frac{\dfrac{\{\sqrt{(1+c)} - \sqrt{c}\}^3}{\sqrt{(1+2c)}}}{z^2 - \{\sqrt{(1+c)} - \sqrt{c}\}^4}\sqrt{[(1+2c)\{\sqrt{(1+c)} - \sqrt{c}\}^2 - z^2]}. \tag{386c}$$

In the numerical case of the parameter $v = \omega_1 + \frac{1}{3}\omega_3$ in §29, we find

$$\varpi - nt = \tfrac{1}{3}\cos^{-1}\frac{-z^2 - 8\sqrt{7}z + 131}{(13 - z^2)^{\frac{3}{2}}}\sqrt{(-z^2 - 2\sqrt{7}z + 5)}$$
$$= \tfrac{1}{3}\sin^{-1}\frac{\sqrt{7}z^2 - 2z - 57\sqrt{7}}{(13 - z^2)^{\frac{3}{2}}}\sqrt{(-z^2 + 2\sqrt{7}z - 3)}. \tag{386d}$$

Or, in the algebraical case, with

$$B = 0,\ N_1 = 0,\ L = (1-c)\sqrt{(1-2c)},$$
$$M = \sqrt{(2c - c^2)}\{1 - c - \sqrt{(1-2c)}\},\ N_2 = \sqrt{(1-2c)},\ N_3 = (1-c)\sqrt{(1-c^2)},$$

we find

$$H = K = \frac{1}{\sqrt{2}}$$

and

$$k^2 = 1 + E = \left\{\frac{1 + \sqrt{(1 - 2c)}}{1 - \sqrt{(1 - 2c)}}\right\}^4,$$

and now the result can be written

$$\varpi - pt = \tfrac{1}{3}\cos^{-1}\frac{z^2 + I_1 z + I_2}{\sqrt{2}\,(k^2 - z^2)^{\frac{3}{2}}}\sqrt{\left(-z^2 + 2\frac{N_2}{M}z + k\right)}$$

$$= \tfrac{1}{3}\sin^{-1}\frac{z^2 - I_1 z + I_2}{\sqrt{2}\,(k^2 - z^2)^{\frac{3}{2}}}\sqrt{\left(-z^2 - 2\frac{N_2}{M}z + k\right)}, \qquad (386e)$$

where

$$I_1 = \frac{N_2 - 3L - (2 - c)(1 - 2c)}{M}, \quad I_2 = -k^{\frac{5}{2}},$$

and then

$$\psi - pt = \tfrac{1}{3}\cos^{-1}\frac{z^2 + H_1 z + H_2}{\sqrt{2}\,(1 - z^2)^{\frac{3}{2}}}\sqrt{\left(-z^2 - 2\frac{N_2}{M}z + k\right)}$$

$$= \tfrac{1}{3}\sin^{-1}\frac{z^2 - H_1 z + H_2}{\sqrt{2}\,(1 - z^2)^{\frac{3}{2}}}\sqrt{\left(-z^2 + 2\frac{N_2}{M}z + k\right)}, \qquad (386f)$$

where

$$H_1 = \frac{\sqrt{(1 - 2c)}}{\sqrt{(2c - c^2)}}\{1 - \sqrt{(1 - 2c)}\}; \quad H_2 = -\frac{1 - \sqrt{(1 - 2c)}}{1 + \sqrt{(1 - 2c)}}.$$

With a parameter $v = \omega_1 + \frac{2}{3}\omega_3$, and, as in (284$b$),

$$B = 0,\ N_1 = 0;\ k^2 = 1 + E = \left\{\frac{\sqrt{(1 - c^2)} + \sqrt{(1 - 2c)}}{\sqrt{(1 - c^2)} - \sqrt{(1 - 2c)}}\right\},$$

$$\varpi - pt = \tfrac{1}{3}\cos^{-1}\frac{z^2 + I_2}{(k^2 - z^2)^{\frac{3}{2}}}\sqrt{\left\{\left(\frac{N_2 - N_3}{M}\right)^2 - z^2\right\}}$$

$$= \tfrac{1}{3}\sin^{-1}\frac{J_1 z}{(k^2 - z^2)^{\frac{3}{2}}}\sqrt{\left\{\left(\frac{N_2 + N_3}{M}\right)^2 - z^2\right\}}, \qquad (386g)$$

where

$$J_1 = \frac{3L + (1 + c)(1 - 2c)}{M}, \quad I_2 = -k^3\frac{M}{N_2 - N_3}.$$

So also with a parameter $v = \frac{1}{3}\omega_3$, and $B = 0$, $N_3 = 0$,

$$\varpi - pt = \tfrac{1}{3}\cos^{-1}\frac{I_1 z}{(z^2 - k^2)^{\frac{3}{2}}}\sqrt{\left\{-z^2 + \left(\frac{N_1 + N_2}{M}\right)^2\right\}}$$

$$= \tfrac{1}{3}\sin^{-1}\frac{z^2 + J_2}{(z^2 - k^2)^{\frac{3}{2}}}\sqrt{\left\{z^2 - \left(\frac{N_1 - N_2}{M}\right)^2\right\}}, \qquad (386h)$$

where

$$k^2 = 1 - E = \left\{\frac{\sqrt{(1 - c^2)} - \sqrt{(2c - c^2)}}{\sqrt{(1 - c^2)} + \sqrt{(2c - c^2)}}\right\}^2 \text{ etc.}$$

The expression for ϖ in the algebraical case, when the parameter $v = \frac{4}{3}\omega_3$, has been given already in (347) and (349).

So also, corresponding to the algebraical case in (358), we shall find

$$\varpi = \tfrac{1}{5}\cos^{-1}\frac{Iz^5 + I_1z^4 + I_2z^3 + I_3z^2 + I_4 + I_5}{(z^2 - 1 + E)^{\frac{5}{2}}}$$

$$= \tfrac{1}{5}\sin^{-1}\frac{Jz^3 + J_1z^2 + J_2z + J_3}{(z^2 - 1 + E)^{\frac{5}{2}}}\sqrt{Z},$$

or

$$\left(\frac{F}{An}\rho e^{\varpi i}\right)^5 = \left\{\frac{F}{An}(\alpha + \beta i)\right\}^5$$

$$= (Iz^5 + I_1z^4 + \dots + I_5) + i(Jz^3 + J_1z^2 + J_2z + J_3)\sqrt{Z}, \tag{387}$$

where

$$I = K = \frac{(7x+1)\sqrt{\{2(18x-1)(x^2+11x-1)\}}}{(3x-1)^{\frac{5}{2}}},$$

$$J = H = \frac{(13x-1)\sqrt{(9x^3+121x^2+33x+3)}}{(3x-1)^{\frac{5}{2}}}; \tag{388}$$

and then

$$I_1 = 10\frac{L}{M}J = -\frac{5}{\sqrt{6}}\,\frac{(13x-1)\sqrt{(9x^3+121x^2+33x+3)}}{(3x-1)\sqrt{\{(2x+1)(x^2+11x-1)\}}},$$

$$J_1 = 10\frac{L}{M}I = -\frac{5}{\sqrt{6}}\,\frac{(7x+1)\sqrt{\{2(18x-1)\}}}{(3x-1)\sqrt{(2x+1)}},$$

$$I_2 = \frac{3}{5}\,\frac{(31x^2+x-1)\sqrt{\{2(18x-1)\}}}{(3x-1)^{\frac{3}{2}}\sqrt{(x^2+11x-1)}},$$

$$J_2 = \frac{1}{3}\,\frac{(18x-1)(31x^2+x-1)\sqrt{(9x^3+121x^2+33x+3)}}{(2x+1)(x^2+11x-1)(3x-1)^{\frac{3}{2}}},$$

$$I_3 = -\frac{10}{3\sqrt{6}}\,\frac{(122x^4+309x^3-357x^2+66x-3)\sqrt{(9x^3+121x^2+33x+3)}}{(2x+1)^{\frac{3}{2}}(x^2+11x-1)^{\frac{3}{2}}(3x-1)},$$

$$J_3 = -\frac{1}{3\sqrt{6}}\,\frac{(31x^2+x-1)(26x^2+21x-21)\sqrt{\{2(18x-1)\}}}{(3x-1)(x^2+11x-1)(2x+1)^{\frac{3}{2}}},$$

$$I_4 = 0,$$

$$I_5 = -\tfrac{2}{5}I_3(1-E) - 2\frac{B}{M}J_3 \tag{389}$$

$$= \frac{1}{9\sqrt{6}}\,\frac{\sqrt{(9x^3+121x^2+33x+3)}}{(2x+1)^{\frac{5}{2}}(x^2+11x-1)^{\frac{5}{2}}}(5812x^6+31956x^5-29655x^4$$

$$+ 37830x^3 - 13495x^2 + 1596x - 57). \tag{389}$$

The special numerical case of $x = 2$ makes

$$I = 3\sqrt{14},\ I_1 = -\frac{25\sqrt{30}}{5},\ I_2 = \frac{25\sqrt{14}}{3},\ I_3 = -\frac{25\sqrt{30}}{9},\ I_4 = 0,\ I_5 = \frac{25\sqrt{30}}{18}$$

$$J = 5\sqrt{5},\ J_1 = -5\sqrt{21}\ ,\ J_2 = \frac{35\sqrt{5}}{3}\ ,\ J_3 = -\frac{25\sqrt{21}}{9},$$

with
$$\frac{L}{M} = -\frac{\sqrt{6}}{12},\quad \frac{B}{M} = \frac{\sqrt{70}}{12},$$
and
$$Z = -z^4 - \frac{5}{6}z^2 - \frac{\sqrt{105}}{9}z - \frac{5}{18};$$

and these numbers will be found to verify.

39. In some cases a factor of Z, say $z - z_0$, can be obtained; and with

$$Z = az^4 + 4bz^3 + 6cz^2 + 4dz + e, \tag{27}$$

we can take Weierstrass's formula

$$\wp u = \tfrac{1}{2}(az_0^2 + 2bz_0 + c) + \frac{az_0^3 + 3bz_0^2 + 3cz_0 + d}{z - z_0}, \tag{390}$$

and now we put

$$\frac{\sigma - s}{M^2} = \wp v - \wp u, \tag{391}$$

where $\wp v$ has the expression given in (86).

The more general form of T, involving circulation, etc., may now be employed, without introducing additional complication.

It is convenient to put

$$1 + aE = k^2, \tag{392}$$

so that

$$Z = a(z^2 - 1)(z^2 - k^2) - 4\left(\frac{Lz - B}{M}\right)^2; \tag{393}$$

thus $z \mp 1$ is a factor of Z if $B = \pm L$, and $z \mp k$ is a factor if $B = \pm Lk$.

The case of $B = L$ may be supposed to be produced when the body, projected originally as a perfectly centred projectile from a rifled gun, has struck an obstacle, thus setting up gyrations in which the axis OC periodically passes through Oz, as in the corresponding *rosette* curves described by the axis of a top discussed in Professor Klein's paper on "The Stability of a Sleeping Top," in the Bulletin of the American Mathematical Society, Jan. 1897.

40. Now, with $B = L$, and $z - 1$ a factor of Z in equation (390),

$$\wp u = \tfrac{1}{2}(a + c) + \frac{a + 3c + d}{z - 1}, \tag{394}$$

and

$$c = -\tfrac{1}{3}a - \tfrac{1}{6}E - \tfrac{2}{3}\frac{L^2}{M^2}, \tag{395}$$

$$d = 2\frac{BL}{M^2} = 2\frac{L^2}{M^2}; \tag{396}$$

so that

$$\wp v - \wp u = \tfrac{1}{4}E + \frac{\tfrac{1}{2}E}{z - 1}, \tag{397}$$

$$\frac{\sigma - s}{M^2} = \tfrac{1}{4}E\frac{z + 1}{z - 1}; \tag{398}$$

or from (65),

$$\frac{\sigma - s}{\sqrt{-\Sigma}} = -\frac{a}{2L}\,\frac{1 + z}{1 - z}, \tag{399}$$

$$\frac{\sigma - s_a}{\sqrt{-\Sigma}} = -\frac{a}{2L}\,\frac{1 + z_a}{1 - z_a}, \tag{400}$$

$$\frac{s - s_a}{\sqrt{-\Sigma}} = -\frac{a}{L}\,\frac{z - z_a}{(1 - z_a)(1 - z)}, \tag{401}$$

or

$$\frac{s - s_a}{M^2} = \tfrac{1}{2}E\frac{z - z_a}{(1 - z_a)(1 - z)}. \tag{402}$$

But from (393),

$$a(z - z_a)(z - z_\beta)(z - z_\gamma) = a(z + 1)(z^2 - k^2) - 4\frac{L^2}{M^2}(z - 1), \tag{403}$$

so that

$$a(1 - z_a)(1 - z_\beta)(1 - z_\gamma) = 2a(1 - k^2) = -2E, \tag{404}$$

and

$$\frac{\tfrac{1}{4}S}{M^6} = \tfrac{1}{8}E^3\frac{Z}{-a(1 - z_a)(1 - z_\beta)(1 - z_\gamma)(z - 1)^4}$$

$$= \tfrac{1}{16}E^2\frac{Z}{(z - 1)^4},$$

$$\frac{\sqrt{S}}{M^3} = \tfrac{1}{2}E\frac{\sqrt{Z}}{(z - 1)^4}, \tag{405}$$

and

$$\frac{ds}{M^2} = \tfrac{1}{2}E\frac{dz}{(z - 1)^2}, \tag{406}$$

so that, as in (48),

$$M\frac{ds}{\sqrt{S}} = \frac{dz}{\sqrt{Z}}. \tag{407}$$

With $L = B$ in equation (59),

$$\frac{L}{M} = \frac{B}{M} = a\frac{N_1N_2N_3}{LM^2},$$

or

$$L^2M = aN_1N_2N_3, \tag{408}$$

so that, from (63),

$$aL^4M^2 = L^4\left(L^2 + 3M^2\wp v - a\frac{\sqrt{-\Sigma}}{L}\right)$$
$$= aN_1^2N_2^2N_3^2 = (L^2+\sigma-s_1)(L^2+\sigma-s_2)(L^2+\sigma-s_3)$$
$$= L^6 + 3L^4M^2\wp v + \tfrac{1}{2}L^2M^4\wp''v + \tfrac{1}{4}\Sigma, \tag{409}$$

or

$$aL^3\sqrt{-\Sigma} + \tfrac{1}{2}L^2M^4\wp''v + \tfrac{1}{4}\Sigma = 0, \tag{410}$$

so that L must be determined from this cubic equation in a very similar manner to that required in the Spherical Pendulum, as discussed in the Proceedings of the London Math. Society, vol. XXVII, p. 607.

Having determined L, or, in a pseudo-elliptic case, having expressed L, σ, $\sqrt{-\Sigma}$, $\wp v$, $\wp''v$ and P in terms of a single parameter,

$$d\psi = 2\frac{L}{M}\frac{1}{1+z}\frac{dz}{\sqrt{Z}}$$
$$= \frac{2L}{1+z}\frac{ds}{\sqrt{S}}, \tag{411}$$

and

$$\frac{z+1}{z-1} = \frac{\sigma - s}{\frac{1}{4}EM^2} = \frac{L}{\frac{1}{2}a\sqrt{-\Sigma}}(\sigma - s),$$

$$z + 1 = \frac{2L(\sigma - s)}{L(\sigma - s) - \frac{1}{2}a\sqrt{-\Sigma}}, \tag{412}$$

so that

$$d\psi = \frac{L(\sigma - s) - \frac{1}{2}a\sqrt{-\Sigma}}{\sigma - s}\frac{ds}{\sqrt{S}}$$
$$= (L - \tfrac{1}{2}P)\frac{ds}{\sqrt{S}} + \tfrac{1}{2}\frac{P(\sigma \sim s) - \sqrt{-\Sigma}}{\sigma \sim s}\frac{ds}{\sqrt{S}}, \tag{413}$$

$$\psi = \frac{L - \frac{1}{2}P}{M}nt + I(v), \tag{414}$$

and similarly for ϖ, which can be made to depend upon the same integral $I(v)$.

The case of $B = -L$ is the same as the preceding, with z changed into $-z$, so that this case does not require separate treatment.

A reference to equation (14) shows that, with $B = \pm L$,

$$\frac{d\psi_1}{dt} = 0, \quad \text{or} \quad \frac{d\psi_2}{dt} = 0,$$

so that
$$\frac{d\phi}{dt} = \left(1 - \frac{C}{A}\right)n \pm \frac{d\psi}{dt}, \tag{415}$$

and ϕ is now pseudo-elliptic with ψ, as also $u + vi$ and $p + qi$, on reference to (179) and (180).

41. A numerical case will serve to elucidate the preceding theory.

Take $v = \frac{4}{5}\omega_3$, and in (410), $a = -1$, $y = x = -\frac{1}{2}$, when
$$\sqrt{-\Sigma} = x^2 = \tfrac{1}{4}, \quad M^4\wp''v = x^2 - x^3 = \tfrac{3}{8}, \tag{416}$$

and then
$$-\tfrac{1}{4}L^3 + \tfrac{3}{16}L^2 - \tfrac{1}{64} = 0, \tag{417}$$

which is satisfied by $L = -\frac{1}{4}$; and now, in (353),
$$H = 2\sqrt{2}, \quad K = -3. \tag{418}$$

Also in (329),
$$M^2 = \tfrac{1}{4}(1+x)^2 - 2x - L^2 - \frac{x^2}{L} = 2, \tag{419}$$
$$LM^2 = -\tfrac{1}{2}, \quad \frac{L}{M} = -\tfrac{1}{8}\sqrt{2}; \tag{420}$$

and in (65) and (59),
$$E = -\frac{2x^2}{LM^2} = 1, \tag{421}$$
$$BLM = -N_1N_2N_3 = -\frac{1}{8\sqrt{2}}, \tag{422}$$
$$\frac{B}{M} = \frac{1}{4\sqrt{2}} = -\frac{L}{M}, \tag{423}$$

and
$$\begin{aligned} Z &= -z^2(z^2-1) - \tfrac{1}{8}(z+1)^2 \\ &= (z+1)(-z^3 + z^2 - \tfrac{1}{8}z - \tfrac{1}{8}). \end{aligned} \tag{424}$$

We therefore take, from (398) and (405),
$$s = \tfrac{1}{2}\frac{1-z}{1+z}, \tag{425}$$
$$s + x = \frac{-z}{1+z}, \tag{426}$$
$$S = \frac{2Z}{(1+z)^4}; \tag{427}$$

and now in (147),
$$\frac{p}{n} = -\tfrac{1}{4}\sqrt{2}; \tag{428}$$

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so that

$$\frac{d\psi}{dz} = \tfrac{1}{4}\sqrt{2}\,\frac{1}{(1-z)\sqrt{Z}}, \tag{429}$$

$$\frac{d(\psi - pt)}{dz} = \tfrac{1}{4}\sqrt{2}\,\frac{-z}{(1-z)\sqrt{Z}}. \tag{430}$$

Taking the pseudo-elliptic integral (Proc. L. M. S. XXV, p. 214)

$$\begin{aligned} I(\tfrac{4}{5}\omega_3) &= \tfrac{1}{2}\int \tfrac{1}{5}\frac{(1-3x)s - x^2}{s\sqrt{S}}\,ds \\ &= \tfrac{1}{5}\cos^{-1}\frac{(1-3x)s^2 - (2x^2 - x^3)s + x^4}{2s^{\frac{5}{2}}} \\ &= \tfrac{1}{5}\sin^{-1}\frac{s - x^2}{2s^{\frac{5}{2}}}\sqrt{S}, \end{aligned} \tag{431}$$

with $y = x = -\tfrac{1}{2}$, and making use of (425), we find that

$$\begin{aligned} \psi - pt &= \tfrac{2}{5}\cos^{-1}\frac{2\sqrt{2}\,z^2 - \tfrac{9}{4}\sqrt{2}\,z + \tfrac{3}{4}\sqrt{2}}{(1-z)^{\frac{5}{2}}}\sqrt{(z+1)} \\ &= \tfrac{2}{5}\sin^{-1}\frac{-3z+1}{(1-z)^{\frac{5}{2}}}\sqrt{(-z^3 + z^2 - \tfrac{1}{8}z - \tfrac{1}{8})}, \end{aligned} \tag{432}$$

and this, on differentiation, will be found to satisfy (430).

As for equations (165) and (169), they become

$$\frac{F\rho}{An} = -z, \tag{433}$$

and

$$\frac{d\varpi}{dz} = \tfrac{1}{4}\sqrt{2}\,\frac{1+z}{-z\sqrt{Z}}, \tag{434}$$

or

$$\frac{d(\varpi - pt)}{dz} = \tfrac{1}{4}\sqrt{2}\,\frac{1}{-z\sqrt{Z}}; \tag{435}$$

the integral of which, derived from the pseudo-elliptic integral (Proc. L. M. S. XXV, p. 213),

$$\begin{aligned} I(\tfrac{2}{5}\omega_3) &= \tfrac{1}{2}\int \tfrac{1}{5}\frac{(3+x)(s+x) - x}{(s+x)\sqrt{S}} \\ &= \tfrac{1}{5}\cos^{-1}\frac{(3+x)s^2 - (1 - 4x - 2x^2)s + x^2 + x^3}{2(s+x)^{\frac{5}{2}}} \\ &= \tfrac{1}{5}\sin^{-1}\frac{s - 1 + x}{2(s+x)^{\frac{5}{2}}}\sqrt{S}, \end{aligned} \tag{436}$$

by making use of (426), becomes

$$\varpi - pt = \tfrac{2}{5}\cos^{-1}\frac{x^2 - \frac{1}{2}z - \frac{1}{4}}{(-z)^{\frac{5}{2}}}\sqrt{(z+1)}$$

$$= \tfrac{2}{5}\sin^{-1}\frac{\sqrt{2}(z+\frac{1}{2})}{(-z)^{\frac{5}{2}}}\sqrt{(-z^3+z^2-\tfrac{1}{8}z-\tfrac{1}{8})}, \tag{437}$$

which will be found to satisfy (435).

Equations (432) and (437) can also be written

$$\{\tan^2\tfrac{1}{2}\theta e^{(\psi-pt)i}\}^{\frac{5}{2}} = \sqrt{2}\,\frac{2z^2-\frac{9}{4}z+\frac{3}{4}}{(z+1)^2} + i\,\frac{-3z+1}{z+1}\sqrt{\frac{-z^3+z^2-\frac{1}{8}z-\frac{1}{8}}{(z+1)^3}}, \tag{438}$$

$$\left\{\frac{F}{An}\rho e^{(\varpi-pt)i}\right\}^{\frac{5}{2}} = (z^2-\tfrac{1}{2}z-\tfrac{1}{4})\sqrt{(z+1)} + i\sqrt{2}(z+\tfrac{1}{2})\sqrt{(-z^3+z^2-\tfrac{1}{8}z-\tfrac{1}{8})}. \tag{439}$$

Now, in (179),

$$P(u+vi) = -F\sin\theta e^{-\left(1-\frac{C}{A}\right)rti+pti}e^{(\psi-pt)i}, \tag{440}$$

while

$$\{\sin\theta e^{(\psi-pt)i}\}^{\frac{5}{2}} = \sqrt{2}\,(2z^2-\tfrac{9}{4}z+\tfrac{3}{4})(z+1)^3 + i(-3z+1)(z+1)^2\sqrt{Z}. \tag{441}$$

But, while keeping Cr unchanged, C and r may be varied so as to make

$$\left(1-\frac{C}{A}\right)r = p, \tag{442}$$

and now

$$\left\{\frac{P}{F}(u+vi)\right\}^{\frac{5}{2}} = -\sqrt{2}\,(2z^2-\frac{9}{4}z+\tfrac{3}{4})(z+1)^3 + i(3z-1)(z+1^2)\sqrt{Z}, \tag{443}$$

with a similar expression for $p+qi$ obtained from (180).

42. With
$$B - Lk = 0 \tag{444}$$
and
$$1 + aE = \frac{B^2}{L^2}, \tag{445}$$

$$Z = \left(z-\frac{B}{L}\right)\left\{a(z^2-1)\left(z+\frac{B}{L}\right) - 4\frac{L^2}{M^2}\left(z-\frac{B}{L}\right)\right\}. \tag{446}$$

A similar calculation shows that

$$\frac{\sigma - s}{M^2} = \tfrac{1}{4}a\left(\frac{B^2}{L^2}-1\right)\frac{z+\frac{B}{L}}{z-\frac{B}{L}}, \tag{447}$$

and
$$aL^3\sqrt{-\Sigma} - \tfrac{1}{2}L^2\wp''v - \tfrac{1}{4}\Sigma = 0, \tag{448}$$

the same as (410), when L is changed into $-L$.

With $z - \frac{B}{L}$ a factor of Z, $\frac{d\varpi}{dz} = 0$ when $z = \frac{B}{L}$, and the curve (α, β) has a series of cusps.

But with $z - \frac{L}{B}$ a factor of Z, $\frac{d\psi}{dz} = 0$ when $z = \frac{L}{B}$, and the cone described by OC round OC has a series of cuspidal edges; and in this case we find

$$\wp u - \wp w = \tfrac{1}{4} a \left(\frac{L^2}{B^2} - 1\right) \frac{z + \frac{L}{B}}{z - \frac{L}{B}}. \tag{449}$$

43. The integrals are *non-elliptic* when the quartic Z has a pair of equal roots; this will be the case if the body is projected in the direction of its axis OC, with a rotation about OC, as if fired from a rifled gun and perfectly centred.

Denoting the velocity of projection by V, and the angular velocity by r, then in the preceding notation,

$F = RV$; $G = CrRV = CrF$, or $B = L$; $E = 0$;

$$2T = RV^2 + Cr^2 = \frac{F^2}{R} + Cr^2; \tag{450}$$

$$Z = a(z^2 - 1)^2 - 4\frac{L^2}{M^2}(z - 1)^2$$
$$= (z-1)^2 \left\{a(z+1)^2 - 4\frac{L^2}{M^2}\right\}, \tag{451}$$

$$\frac{ndt}{dz} = \frac{1}{(1 - z)\sqrt{\left\{a(1+z)^2 - 4\frac{L^2}{M^2}\right\}}}, \tag{452}$$

$$\frac{d\psi}{dz} = 2\frac{L}{M}\frac{1}{(1 - z^2)\sqrt{\left\{a(1+z)^2 - 4\frac{L^2}{M^2}\right\}}}. \tag{453}$$

If the body is oblate and $a = -1$, the only solution is $z = 1$, so that this motion is stable; but, at the same time, putting

$$1 - z = y^2, \tag{454}$$

then
$$\frac{dy^2}{dt^2} = n^2 y^2 \left(-1 - \frac{L^2}{M^2} + y^2 - \tfrac{1}{4} y^4\right), \tag{455}$$

so that
$$\frac{d^2y}{dt^2} = -n^2\left(1+\frac{L^2}{M^2}\right)y + \dots.$$
$$\simeq -n^2\left(1+\frac{L^2}{M^2}\right)y, \tag{456}$$

when y is small; so that the number of complete oscillations in one second is
$$\frac{n}{2\pi}\sqrt{\left(1+\frac{L^2}{M^2}\right)}. \tag{457}$$

But in a prolate body, with $a = +1$, then (i) with $\frac{L}{M} > 1$,
$$n\frac{dt}{dz} = \frac{1}{(1-z)\sqrt{\left\{(1+z)^2 - \frac{L^2}{M^2}\right\}}}; \tag{458}$$

and integrating,
$$nt = \frac{1}{\sqrt{\left(\frac{L^2}{M^2}-1\right)}}\cos^{-1}\sqrt{\frac{\left(\frac{L}{M}+1\right)\left(1+z-2\frac{L}{M}\right)}{2\frac{L}{M}(1-z)}}$$
$$= \frac{1}{\sqrt{\left(\frac{L^2}{M^2}-1\right)}}\sin^{-1}\sqrt{\frac{\left(\frac{L}{M}-1\right)\left(1+z+2\frac{L}{M}\right)}{2\frac{L}{M}(1-z)}}, \tag{459}$$

but this does not represent any real motion of the axis, since z oscillates between ± 1.

The only solution is therefore $z = 1$; and to find the period of a small oscillation, putting
$$1 - z = y^2 \tag{454}$$

as before,
$$\frac{dy^2}{dt^2} = n^2y^2\left(1-\frac{L^2}{M^2} - y^2 + \tfrac{1}{4}y^4\right), \tag{455}$$
$$\frac{d^2y}{dt^2} = -n^2\left(\frac{L^2}{M^2}-1\right)y + \dots, \tag{456}$$

and the axis thus makes
$$\frac{n}{2\pi}\sqrt{\left(\frac{L^2}{M^2}-1\right)} \tag{457}$$

complete oscillations per second.

Thus to secure the stability of this elongated projectile in its flight we must make

$$\frac{L}{M} = \frac{Cr}{2An} > 1. \tag{458}$$

44. But (ii) with $\frac{L}{M} < 1$, integrating

$$nt = \frac{1}{\sqrt{\left(1 - \frac{L^2}{M^2}\right)}} \operatorname{ch}^{-1} \sqrt{\frac{\left(1 - \frac{L}{M}\right)\left(1 + z + 2\frac{L}{M}\right)}{2\frac{L}{M}(1 - z)}}$$

$$= \frac{1}{\sqrt{\left(1 - \frac{L^2}{M^2}\right)}} \operatorname{sh}^{-1} \sqrt{\frac{\left(1 + \frac{L}{M}\right)\left(1 + z - 2\frac{L}{M}\right)}{2\frac{L}{M}(1 - z)}}, \tag{459}$$

so that the motion of the axis is unstable, and, after an infinite time, is given by the equation (459) above.

Putting
$$\frac{L}{M} = \cos\alpha, \tag{460}$$

$$\operatorname{ch}^2(nt\sin\alpha) = \frac{(1 - \cos\alpha)(1 + z + 2\cos\alpha)}{2\cos\alpha(1 - z)}, \tag{461}$$

$$\operatorname{sh}^2(nt\sin\alpha) = \frac{(1 + \cos\alpha)(1 + z - 2\cos\alpha)}{2\cos\alpha(1 - z)}, \tag{462}$$

$$\operatorname{ch}(2nt\sin\alpha) = \frac{z - \cos 2\alpha}{\cos\alpha(1 + z)}, \tag{463}$$

$$z = \frac{\cos 2\alpha + \cos\alpha \operatorname{ch}(2nt\sin\alpha)}{1 + \cos\alpha \operatorname{ch}(2nt\sin\alpha)}, \tag{464}$$

$$1 + z = 2\cos\alpha \frac{\cos\alpha + \operatorname{ch}(2nt\sin\alpha)}{1 + \cos\alpha \operatorname{ch}(2nt\sin\alpha)}; \tag{465}$$

$$\frac{d\psi}{dt} = \frac{2n\cos\alpha}{1 + z}$$

$$= n\frac{1 + \cos\alpha \operatorname{ch}(2nt\sin\alpha)}{\cos\alpha + \operatorname{ch}(2nt\sin\alpha)}$$

$$= n\cos\alpha + \frac{n\sin^2\alpha}{\cos\alpha + \operatorname{ch}(2nt\sin\alpha)}, \tag{466}$$

$$\psi = nt\cos\alpha + \tfrac{1}{2}\cos^{-1}\frac{1 + \cos\alpha \operatorname{ch}(2nt\sin\alpha)}{\cos\alpha + \operatorname{ch}(2nt\sin\alpha)}$$

$$= nt\cos\alpha + \tfrac{1}{2}\cos^{-1}\frac{2\cos\alpha}{1 + z}. \tag{467}$$

At the same time, from (169),

$$\frac{d\varpi}{dt} = 2n\frac{L}{M}\frac{z - z^2}{1 - z^2} = 2n\cos\alpha\frac{z}{1+z}$$

$$= n\frac{\cos 2\alpha + \cos\alpha \operatorname{ch}(2nt\sin\alpha)}{\cos\alpha + \operatorname{ch}(2nt\sin\alpha)}$$

$$= n\cos\alpha - \frac{n\sin^2\alpha}{\cos\alpha \operatorname{ch}(2nt\sin\alpha)}, \tag{468}$$

$$\varpi = nt\cos\alpha - {}^{1}\cos^{-1}\frac{1 + \cos\alpha \operatorname{ch}(2nt\sin\alpha)}{\cos\alpha + \operatorname{ch}(2nt\sin\alpha)}$$

$$= nt\cos\alpha - \tfrac{1}{2}\cos^{-1}\frac{2\cos\alpha}{1+z}; \tag{469}$$

so that, as in (167),

$$\psi - \varpi = \cos^{-1}\frac{2\cos\alpha}{1+z} = \sin^{-1}\frac{\sqrt{\{(1+z)^2 - 4\cos^2\alpha\}}}{1+z}. \tag{470}$$

Since $E = 0$, $a = 1$,

$$\frac{F^2\rho^2}{A^2n^2} = 1 - z^2, \tag{471}$$

so that the (α, β) or (ρ, ϖ) curve is given by

$$\varpi = nt\cos\alpha - \tfrac{1}{2}\cos^{-1}\frac{2\cos\alpha}{1 + \sqrt{\left(1 - \dfrac{F^2\rho^2}{A^2n^2}\right)}}$$

$$= nt\cos\alpha - \tfrac{1}{2}\cos^{-1}\frac{2\cos\alpha\left\{1 - \sqrt{\left(1 - \dfrac{F^2\rho^2}{A^2n^2}\right)}\right\}}{\dfrac{F^2\rho^2}{A^2n^2}}. \tag{472}$$

From equation (155),

$$F\frac{d\gamma}{dt} = \frac{F^2}{P} + An^2z^2$$

$$= \frac{F^2}{P} + An^2\left\{1 - \frac{2\sin^2\alpha}{1 + \cos\alpha \operatorname{ch}(2nt\sin\alpha)}\right\}^2, \tag{473}$$

or

$$\frac{d\gamma}{dt} = V - 4\frac{An^2}{F}\frac{\sin^2\alpha}{1 + \cos\alpha \operatorname{ch}(2nt\sin\alpha)}$$

$$+ 4\frac{An^2}{F}\frac{\sin^4\alpha}{\{1 + \cos\alpha \operatorname{ch}(2nt\sin\alpha)\}}, \tag{474}$$

and integrating,

$$\gamma = Vt - \frac{An}{F}\frac{\sin 2\alpha \operatorname{sh}(2nt\sin\alpha)}{1 + \cos\alpha \operatorname{ch}(2nt\sin\alpha)}, \tag{475}$$

thus determining α, β, γ as functions of the time t, an infinite time after the start.

If
$$L=0, \quad \cos\alpha=0, \quad \alpha=\tfrac{1}{2}\pi, \tag{476}$$

as if the body was fired from a smooth-bore gun without rotation; then

$$\frac{d\psi}{dt}=0, \quad \frac{d\varpi}{dt}=0,$$

while
$$\frac{dz}{dt}=n(1+z^2), \tag{477}$$

$$z=\tanh nt, \tag{478}$$

$$\rho=\frac{An}{F}\operatorname{sech} nt, \tag{479}$$

and the motion is confined to one plane.

45. When the body is projected sideways, in the direction OA, with velocity V, and with component angular velocities p and r about OA and OC, then initially,

$$u=V, \quad v=0, \quad w=0, \quad q=0, \tag{480}$$

so that
$$F=PV, \quad G=PVAp,$$

$$\frac{p}{n}=\frac{G}{AnF}=2\frac{L}{M}; \tag{481}$$

$$\begin{aligned}2T-Cr^2-\frac{F^2}{R}&=PV^2+Ap^2+Cr^2-\frac{P^2}{R}V^2-Cr^2\\&=-F^2\left(\frac{1}{R}-\frac{1}{P}\right)V^2+Ap^2\\&=-An^2a+4An^2\frac{L^2}{M^2};\end{aligned} \tag{482}$$

$$D=-a+4\frac{L^2}{M^2}, \quad E=-a+4\frac{B^2}{M^2}; \tag{483}$$

$$\begin{aligned}Z&=a(z^2-1)^2+a(z^2-1)-4\frac{L^2}{M^2}(z^2-1)-4\left(\frac{Bz-L}{M}\right)^2\\&=z\left\{az(z^2-1)-4\frac{B^2+L^2}{M^2}z+8\frac{BL}{M}\right\};\end{aligned} \tag{484}$$

so that this case comes under the head of those cases where a factor of Z is known; and here we have to put

$$\begin{aligned}\frac{\sigma-s}{M^2}&=\wp v-\wp u\\&=\tfrac{1}{4}a+\frac{B^2}{M^2}+\frac{BL}{M^2 z}.\end{aligned} \tag{485}$$

When there is no rotation about OC, so that $r = 0$, $B = 0$,

$$Z = z^2 \left\{ a (z^2 - 1) - 4 \frac{L^2}{M^2} \right\}, \tag{486}$$

$$\frac{n\,dt}{dz} = \frac{1}{z \sqrt{\left\{ a (z^2 - 1) - 4 \dfrac{L^2}{M^2} \right\}}}, \tag{487}$$

and therefore $z = 0$ is the only solution for a prolate body, with $a = 1$, since $z^2 - 1$ is negative; but

$$\frac{d^2 z}{dt^2} \simeq - n^2 \left(1 + 4 \frac{L^2}{M^2} \right) z, \tag{488}$$

so that the axis makes

$$\frac{n}{2\pi} \sqrt{\left(1 + 4 \frac{L^2}{M^2} \right)} \tag{489}$$

small oscillations a second.

But with an oblate body, and $a = -1$,

$$n \frac{dt}{dz} = \frac{1}{z \sqrt{\left(1 - 4 \dfrac{L^2}{M^2} - z^2 \right)}}; \tag{490}$$

so that the form of the integral is different according as $1 - 4 \dfrac{L^2}{M^2}$ is positive or negative.

When $1 - 4 \dfrac{L^2}{M^2}$ is negative, the only real solution is again $z = 0$, with a small nutation of the axis OC, making

$$\frac{n}{2\pi} \sqrt{\left(4 \frac{L^2}{M^2} - 1 \right)} \tag{491}$$

oscillations a second.

But, with $1 - 4 \dfrac{L^2}{M^2}$ positive,

$$nt = \frac{1}{\sqrt{\left(1 - 4 \dfrac{L^2}{M^2} \right)}} \operatorname{ch}^{-1} \frac{\sqrt{\left(1 - 4 \dfrac{L^2}{M^2} \right)}}{z}$$

$$z = \frac{\sqrt{\left(1 - 4 \dfrac{L^2}{M^2} \right)}}{\operatorname{ch} \sqrt{\left(1 - 4 \dfrac{L^2}{M^2} \right)} nt}. \tag{492}$$

Also

$$\frac{d\psi}{dz} = 2\frac{L}{M}\frac{1}{z(1-z^2)\sqrt{\left(1-4\frac{L^2}{M^2}-z^2\right)}}$$

$$= 2\frac{L}{M}\left(\tfrac{1}{z}+\frac{z}{1-z^2}\right)\frac{1}{\sqrt{\left(1-4\frac{L^2}{M^2}-z^2\right)}}$$

$$= 2\frac{L}{M}n\frac{dt}{dz} + 2\frac{L}{M}\frac{z}{(1-z^2)\sqrt{\left(1-4\frac{L^2}{M^2}-z^2\right)}}, \tag{493}$$

$$\psi = 2\frac{L}{M}nt + \sin^{-1}\frac{2\frac{L}{M}}{\sqrt{(1-z^2)}}$$

$$= 2\frac{L}{M}nt + \cos^{-1}\frac{\sqrt{\left(1-4\frac{L^2}{M^2}-z^2\right)}}{\sqrt{(1-z^2)}}. \tag{494}$$

With $B=0$, $E=1$, $a=-1$, and in (165),

$$\frac{F^2\rho^2}{A^2n^2} = z^2 - 1 + E = z^2, \tag{495}$$

$$\rho = \frac{An}{F}z = \frac{An}{F}\sqrt{\left(1-4\frac{L^2}{M^2}\right)}\operatorname{sech}\sqrt{\left(1-4\frac{L^2}{M^2}\right)}nt, \tag{496}$$

and from (169),

$$\frac{d\varpi}{dz} = 2\frac{L}{M}\frac{1}{\sqrt{Z}} = 2\frac{L}{M}\frac{ndt}{dz}, \tag{497}$$

$$\varpi = 2\frac{L}{M}nt, \tag{498}$$

so that

$$\rho = \frac{An}{F}\sqrt{\left(1-4\frac{L^2}{M^2}\right)}\operatorname{sech}\frac{\sqrt{\left(1-4\frac{L^2}{M^2}\right)}}{2\frac{L}{M}}\varpi, \tag{499}$$

a curve of the nature of a *separating* herpolhode.

At the same time, in (155),

$$F\frac{d\gamma}{dt} = \frac{F^2}{P} - An^2z^2$$

$$\frac{d\gamma}{dt} = V - \frac{An^2}{F}\left(1-4\frac{L^2}{M^2}\right)\operatorname{sech}^2\sqrt{\left(1-4\frac{L^2}{M^2}\right)}nt, \tag{500}$$

$$\gamma = Vt - \frac{An}{F}\sqrt{\left(1-4\frac{L^2}{M^2}\right)}\tanh\sqrt{\left(1-4\frac{L^2}{M^2}\right)}nt, \tag{501}$$

thus determining the motion of the body completely.

46. The discussion proceeds in the same manner when we put

$$p=0, \quad G=0, \quad L=0, \tag{502}$$

and now

$$Z=z^2\left\{a(z^2-1)-4\frac{B_2}{M^2}\right\}, \tag{503}$$

$$\frac{dnt}{dz}=\frac{1}{z\sqrt{\left\{a(z^2-1)-4\frac{B^2}{M^2}\right\}}}, \tag{504}$$

$$\begin{aligned}\frac{d\psi}{dz}&=-2\frac{B}{M}\frac{z}{(1-z^2)\sqrt{Z}}\\&=-2\frac{B}{M}\frac{1}{(1-z^2)\sqrt{\left\{a(z^2-1)-4\frac{B^2}{M^2}\right\}}};\end{aligned} \tag{505}$$

and since

$$E=-a+4\frac{B^2}{M^2}, \tag{506}$$

$$\begin{aligned}\frac{d\varpi}{dz}&=2\frac{B}{M}\frac{z}{(1+aE-z^2)\sqrt{Z}}\\&=2\frac{B}{M}\frac{1}{\left(4a\frac{B^2}{M^2}-z^2\right)\sqrt{\left\{a(z^2-1)-4\frac{B^2}{M^2}\right\}}}.\end{aligned} \tag{507}$$

Considering only the case of the oblate body, $a=-1$, with $1-4\frac{B^2}{M^2}$ positive,

$$z=\sqrt{\left(1-4\frac{B^2}{M^2}\right)}\operatorname{sech}\sqrt{\left(1-4\frac{B^2}{M^2}\right)}nt, \tag{508}$$

$$\begin{aligned}\psi&=\cos^{-1}\frac{2\frac{B}{M}z}{\sqrt{\left(1-4\frac{B^2}{M^2}\right)}\sqrt{(1-z^2)}}\\&=\sin^{-1}\frac{\sqrt{\left(1-4\frac{B^2}{M^2}-z^2\right)}}{\sqrt{\left(1-4\frac{B^2}{M^2}\right)}\sqrt{(1-z^2)}},\end{aligned} \tag{509}$$

$$\begin{aligned}\varpi&=\cos^{-1}\frac{z}{\sqrt{\left(1-4\frac{B^2}{M^2}\right)}\sqrt{\left(4\frac{B^2}{M^2}+z^2\right)}}\\&=\sin^{-1}\frac{2\frac{B}{M}\sqrt{\left(1-4\frac{B^2}{M^2}-z^2\right)}}{\sqrt{\left(1-4\frac{B^2}{M^2}\right)}\sqrt{\left(4\frac{B^2}{M^2}+z^2\right)}};\end{aligned} \tag{510}$$

$$\frac{F^2\rho^2}{A^2n^2}=4\frac{B^2}{M^2}+z^2; \tag{511}$$

so that

$$\sqrt{\left(1-4\frac{B^2}{M^2}\right)}\frac{F\rho}{An}\sin\varpi=2\frac{B}{M}\sqrt{\left(1-\frac{F^2\rho^2}{A^2n^2}\right)}, \tag{512}$$

or, putting $\frac{An}{F}=c$,

$$\frac{\alpha^2}{c^2}+\tfrac{1}{4}\frac{M^2}{B^2}\frac{\beta^2}{c^2}=1, \tag{513}$$

the equation of an ellipse.

The expression for V will be of the same nature as in (475).

This last state of motion can be illustrated experimentally by holding a disc of cardboard between the finger and thumb and flicking it sideways in the air.

47. The analysis developed here for the motion of Solid of a Revolution under No Forces in Infinite Frictionless Liquid is also available for other dynamical problems; for instance, in the exact treatment of the Precession and Nutation of the Earth's Axis, when, as in Poinsot's "Addition à la connaisance des temps," 1858, the mass of the disturbing Moon and Sun is supposed distributed in the form of a circular band or ring in the plane of the ecliptic, or else condensed into two equal repelling spheres, placed at the opposite poles of ecliptic at the same radial distance.

These equations are considered by Tisserand in the Comptes rendus, t. 101, 1885; as also Gylden's intermediate orbit, described under a central force varying partly as the distance and partly inversely as the square of the distance, which leads to similar analysis.

The motion of a ball rolling on a gravitating sphere, in which variations of internal density give rise to zonal harmonics of the first and second order in the expression of the potential, as well as the motion of a particle sliding on the smooth surface of a homogeneous gravitating ellipsoid of revolution, are further applications of the same analysis.

48. Taking the case of Precession and Nutation, in which the disturbance is due to a mass M, distributed in the form of a ring of radius R, the forces acting upon the Earth, an oblate spheroid of which the equatorial and polar moments of inertia are denoted by A and C, are equivalent to a couple round the line of nodes, of moment

$$\frac{3\gamma M}{R^3}(C-A)\sin\vartheta\cos\vartheta, \tag{514}$$

tending to decrease ϑ, where ϑ denotes the obliquity of the ecliptic, and γ the constant of gravitation; so that ψ denoting the longitude of the node, the equations of Energy and Momentum may be written

$$\tfrac{1}{2}A\frac{d\vartheta^2}{dt^2}+\tfrac{1}{2}A\sin^2\vartheta\frac{d\psi^2}{dt^2}=-\tfrac{3}{2}\frac{\gamma M}{R^3}(C-A)\sin^2\vartheta+H, \tag{515}$$

$$A\sin^2\vartheta\frac{d\psi}{dt}+Cr\cos\vartheta=G, \tag{516}$$

where H and K are the constants of the problem, and r the constant angular velocity of the Earth about its axis.

Putting $\cos\vartheta=z$ and eliminating $\dfrac{d\psi}{dt}$ between (515) and (516),

$$\begin{aligned}\frac{dz^2}{dt^2}&=-3\frac{\gamma M}{R^3}\frac{C-A}{A}(1-z^2)^2+\frac{2H}{A}(1-z^2)-\left(\frac{G-Crz}{A}\right)^2\\&=n^2\left\{-(z^2-1)(z^2-1+D)-\left(\frac{Crz-G}{An}\right)^2\right\}\\&=n^2Z,\end{aligned} \tag{517}$$

on putting

$$\frac{3\gamma M}{R^3}\frac{C-A}{A}=n^2,\quad \frac{2H}{A}=n^2D, \tag{518}$$

and then

$$\frac{d\psi}{dt}=\frac{G-Crz}{A\sin^2\vartheta}, \tag{519}$$

$$\frac{d\psi}{dz}=\frac{G-Crz}{An(1-z^2)\sqrt{Z}}, \tag{520}$$

equations of exactly the same form as those required for the motion of an oblate solid in liquid, so that the previous pseudo-elliptic solutions are immediately available; for instance, the algebraical solutions given by equations (345) and (358).

To gain some idea of the actual magnitude of the constants in this problem in the actual case of the Earth, let μ denote the mean angular velocity of Precession and ω the mean obliquity of the ecliptic, then (Quarterly Journal of Mathematics, vol. XIV, p. 173)

$$\mu=\frac{3\gamma M}{2R^3}\frac{C-A}{Cr}\cos\omega, \tag{521}$$

so that

$$n^2=2\frac{C}{A}r\mu\sec\omega; \tag{522}$$

and in this equation we may take

$$\frac{C}{A}=1, \quad \omega=23^\circ\ 27', \tag{523}$$

and, with a year as unit of time, and an annual Precession of 50″.25,

$$r=2\pi\times 366, \quad \mu=\frac{50.25}{206265}; \tag{524}$$

this makes

$$n=1.115 \text{ and } \frac{Cr}{An}=2062. \tag{525}$$

But if we try to utilize the algebraical case given by equation (345) we find from (343),

$$\frac{Cr}{An}=2\frac{B}{M}=\frac{H}{K}=\frac{H}{\sqrt{(H^2+1)}}, \tag{526}$$

which, with $H>1$, must be <0.707, so that the angular velocity of the Earth would require to be reduced to about one three-thousandth of its present amount for this motion to be possible; and now the path of the pole in the sky would give rise to interesting astronomical speculations.

49. In Gylden's orbit, with polar coordinates $\rho=\frac{1}{u}$ and ϖ, the central force

$$P=\mu\rho+\frac{\mu'}{\rho^2}; \tag{527}$$

or, more generally,

$$P=a\rho+2b-\frac{4d}{\rho^2}, \tag{528}$$

suppose; so that

$$\tfrac{1}{2}\frac{d\rho^2}{dt^2}+\tfrac{1}{2}\rho^2\frac{d\varpi^2}{dt^2}=\tfrac{1}{2}a\rho^2+2b\rho+3c+\frac{4d}{\rho}, \tag{529}$$

and

$$\rho^2\frac{d\varpi}{dt}=h. \tag{530}$$

Thence

$$\begin{aligned}\rho^2\frac{d\rho^2}{dt^2}&=a\rho^4+4b\rho^3+6c\rho^2+4d\rho-h^2,\\ &=R,\end{aligned} \tag{531}$$

suppose; so that

$$\frac{d\varpi}{d\rho}=\frac{h}{\rho\sqrt{R}}, \text{ or } \frac{d\varpi}{du}=\frac{-hu}{\sqrt{(-h^2u^4+4du^3+6cu^2+4bu+a)}} \tag{531a}$$

is the differential relation for the orbit, and thus pseudo-elliptic cases can be constructed, immediately from Abel's analysis (*Œuvres complètes*).

50. For the motion of a ball, of radius b, rolling on a sphere of radius $R-b$, under a zonal potential denoted by V, the equations of Energy and Momentum become

$$\tfrac{1}{2}\left(1+\frac{k^2}{b^2}\right)R^2\left(\frac{d\vartheta^2}{dt^2}+\sin^2\vartheta\frac{d\psi^2}{dt^2}\right)=V+H, \tag{532}$$

$$\left(1+\frac{k^2}{b^2}\right)\sin^2\vartheta\frac{d\psi}{dt}+\frac{k^2}{bR}r\cos\vartheta=G, \tag{533}$$

where k denotes the radius of gyration of the ball.

The variable part of V depending on $\cos\vartheta$ and $\cos^2\vartheta$ or $\sin^2\vartheta$, we can bring our equations into the same shape as before by putting

$$V+H=\tfrac{1}{2}n^2R^2\left(1+\frac{k^2}{b^2}\right)(a\sin^2\vartheta+4b'\cos\vartheta+D), \tag{534}$$

and

$$G=2n\left(1+\frac{k^2}{b^2}\right)\frac{L}{M},\quad \frac{k^2r}{bR}=2n\left(1+\frac{k^2}{b^2}\right)\frac{B}{M}, \tag{535}$$

so that

$$\frac{d\vartheta^2}{dt^2}+\sin^2\vartheta\frac{d\psi^2}{dt^2}=n^2(a\sin^2\vartheta+4b'\cos\vartheta+D), \tag{536}$$

$$\sin^2\vartheta\frac{d\psi}{dt}=2n\frac{L-B\cos\vartheta}{M}. \tag{537}$$

Eliminating $\frac{d\psi}{dt}$ and putting $\cos\vartheta=z$, we obtain as before

$$\frac{dz}{dt}=n\sqrt{Z},$$

where

$$Z=a(1-z^2)^2+(4b'z+D)(1-z^2)-4\left(\frac{L-Bz}{M}\right)^2 \tag{538}$$

and

$$\frac{d\psi}{dz}=2\frac{L-Bz}{M(1-z^2)\sqrt{Z}}, \tag{539}$$

of which the special case for $b'=0$ has received consideration, and the solution proceeds as formerly.

When the ball is projected without any spin, so that $r=0$, the equations of motion are the same as for a particle on a smooth sphere, in the same field of force; and now, with $b'=0$,

$$Z=a(1-z^2)(1-z^2+aD)-4\frac{L^2}{M^2}. \tag{540}$$

With a positive and equal to $+1$, we can put

$$\frac{dz^2}{dt^2} = n^2 (\alpha^2 - z^2)(\beta^2 - z^2), \tag{541}$$

and taking

$$\beta > 1 > \alpha > z > -\alpha > -1 > \beta, \tag{542}$$

the particle or ball crosses the equator, and

$$nt = \int \frac{dz}{\sqrt{\{(\alpha^2 - z^2)(\beta^2 - z^2)\}}}$$

$$= \frac{1}{\beta} \operatorname{sn}^{-1}\left(\frac{z}{\alpha}, \frac{\alpha}{\beta}\right), \tag{543}$$

or

$$z = \alpha \operatorname{sn} \beta nt. \tag{544}$$

Putting $z^2 = 1$ makes

$$4\frac{L^2}{M^2} = (\beta^2 - 1)(1 - \alpha^2), \tag{545}$$

so that

$$\frac{d\psi}{dz} = \frac{\sqrt{\{(\beta^2 - 1)(1 - \alpha^2)\}}}{(1 - z^2)\sqrt{\{(\alpha^2 - z^2)(\beta^2 - z^2)\}}}. \tag{546}$$

Now if we put

$$\frac{z^2}{\alpha^2} = x \text{ and } \frac{1}{\alpha^2} = a, \quad k^2 = \frac{\alpha^2}{\beta^2}; \tag{547}$$

$$\psi = \tfrac{1}{2} \int \frac{\sqrt{\{-a(1-a)(1-k^2a)\}}}{a - x} \frac{dx}{\sqrt{\{x(1-x)(1-k^2x)\}}}$$

$$= \tfrac{1}{2} \int \frac{\sqrt{(-A)}\, dx}{(a - x)\sqrt{X}}, \tag{548}$$

where

$$X = x(1 - x)(1 - k^2x), \tag{549}$$

and A is the value of X when a replaces x; this is a canonical form of the Elliptic Integral of the Third Kind, with the appropriate multiplier $\sqrt{(-A)}$ in the numerator.

With a negative and equal to -1, we can have

$$Z = (\alpha^2 - z^2)(z^2 \mp \beta^2). \tag{550}$$

With

$$Z = (\alpha^2 - z^2)(z^2 - \beta^2), \tag{551}$$

the path of the particle is bounded by two parallels of latitude on the same side of the equator, and

$$1 > \alpha > z > \beta > 0 > -\beta > z > -\alpha > 1; \tag{552}$$

$$nt = \int \frac{dz}{\sqrt{\{(\alpha^2 - z^2)(z^2 - \beta)^2\}}}$$

$$= \frac{1}{\alpha} \,\mathrm{dn}^{-1} \left\{ \frac{z}{\alpha}, \sqrt{\left(1 - \frac{\beta^2}{\alpha^2}\right)} \right\}, \tag{553}$$

$$z = \alpha \,\mathrm{dn}\, \alpha nt, \tag{554}$$

and

$$4 \frac{L^2}{M^2} = (1 - \alpha^2)(1 - \beta^2), \tag{555}$$

$$\frac{d\psi}{dz} = \frac{\sqrt{\{(1 - \alpha^2)(1 - \beta^2)\}}}{(1 - z^2)\sqrt{\{(\alpha^2 - z^2)(z^2 - \beta^2)\}}}. \tag{556}$$

Putting

$$\frac{z^2}{\alpha^2} = x, \quad \frac{1}{\alpha^2} = a, \quad 1 - \frac{\beta^2}{\alpha^2} = k^2, \tag{557}$$

makes

$$\psi = \tfrac{1}{2} \int \frac{\sqrt{(-A)}\, dx}{(a - x)\sqrt{X}}, \tag{558}$$

where

$$X = x(1 - x)(x - k'^2). \tag{559}$$

With

$$Z = (\alpha^2 - z^2)(z^2 + \beta^2), \tag{560}$$

the particle again crosses the equator, and

$$nt = \int \frac{dz}{\sqrt{\{(\alpha^2 - z^2)(z^2 + \beta^2)\}}}$$

$$= \frac{1}{\sqrt{(\alpha^2 + \beta^2)}} \,\mathrm{cn}^{-1} \left\{ \frac{z}{\alpha}, \frac{\alpha}{\sqrt{(\alpha^2 + \beta^2)}} \right\}, \tag{561}$$

$$z = \alpha \,\mathrm{cn}\, \sqrt{(\alpha^2 + \beta^2)}\, nt, \tag{562}$$

$$4 \frac{L^2}{M^2} = (1 - \alpha^2)(1 + \beta^2), \tag{563}$$

and with

$$\frac{z^2}{\alpha^2} = x, \quad \frac{1}{\alpha^2} = a, \quad \frac{\alpha^2}{\alpha^2 + \beta^2} = k^2, \tag{564}$$

$$\psi = \tfrac{1}{2} \int \frac{\sqrt{(-A)}\, dx}{(a - x)\sqrt{X}}, \tag{565}$$

where

$$X = x(1 - x)\left(x + \frac{k'^2}{k^2}\right). \tag{566}$$

Thence pseudo-elliptic cases can easily be constructed, but the secular term cannot be cancelled.

But when the ball is projected with appropriate spin r, we have an additional constant at our disposal, and it is possible to make the path an algebraical curve; thus, for instance, from (345), the equations

$$\sin^3\theta \cos 3\psi = H \cos^3\theta - 1, \tag{567}$$

$$\sin^3\theta \sin 3\psi = \cos\theta \sqrt{\{-(H^2+1)\cos^4\theta + 3\cos^2\theta + 2H\cos\theta - 3\}}, \tag{568}$$

represent a possible algebraical trajectory of the centre of the ball, rolling on this gravitating sphere, the circumstances of the initial projection being chosen appropriately. So also equations (358).

51. In the motion of a particle on a smooth, homogeneous, gravitating ellipsoid, bounded by the surface of revolution about $O\gamma$ of an ellipse

$$\frac{\rho^2}{\alpha^2} + \frac{\gamma^2}{\beta^2} = 1, \tag{569}$$

referred to $O\rho$ and $O\gamma$ as coordinate axes, the variable part of the potential on the surface may be put equal to $\frac{1}{2}A\rho^2$, so that, with cylindrical coordinates ρ, ϖ, γ, the equations of Momentum and Energy may be written

$$\rho^2 \frac{d\varpi}{dt} = K, \tag{570}$$

$$\frac{d\rho^2}{dt^2} + \rho^2 \frac{d\varpi^2}{dt^2} + \frac{d\gamma^2}{dt^2} = A\rho^2 + B. \tag{571}$$

From (569) we find

$$\frac{d\gamma}{dt} = \frac{\beta}{\alpha} \frac{\rho}{\sqrt{(\alpha^2-\rho^2)}} \frac{d\rho}{dt}, \tag{572}$$

so that

$$\frac{d\rho^2}{dt^2} + \frac{K^2}{\rho^2} + \frac{\beta^2}{\alpha^2} \frac{\rho^2}{\alpha^2-\rho^2} \frac{d\rho^2}{dt^2} = A\rho^2 + B,$$

or

$$\left\{\alpha^2 - \left(1 - \frac{\beta^2}{\alpha^2}\right)\rho^2\right\}\rho^2 \frac{d\rho^2}{dt^2} = (\alpha^2-\rho^2)(A\rho^4 + B\rho^2 - K^2). \tag{573}$$

Putting $\rho^2 = \alpha^2 z$,

$$\frac{dz^2}{dt^2} = 4\frac{(1-z)\left(Az^2 + \frac{B}{\alpha^2}z - \frac{K^2}{\alpha^4}\right)}{1 - \left(1 - \frac{\beta^2}{\alpha^2}\right)z}$$

$$= 4\frac{Z}{\left\{1 - \left(1 - \frac{\beta^2}{\alpha^2}\right)z\right\}^2}, \tag{574}$$

where

$$Z = (z-1)\left\{\left(1-\frac{\beta^2}{\alpha^2}\right)z-1\right\}\left(Az^2+\frac{B}{\alpha^2}z-\frac{K^2}{\alpha^4}\right) \tag{575}$$

and

$$z\frac{d\varpi}{dt} = \frac{K}{\alpha^2}, \tag{576}$$

so that

$$\frac{d\varpi}{dz} = \tfrac{1}{2}\frac{K}{\alpha^2}\,\frac{1-\left(1-\frac{\beta^2}{\alpha^2}\right)z}{z\sqrt{Z}}, \tag{577}$$

and this can be treated in the same manner as the previous equations (531) for Gylden's orbit.

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